RENORMALIZATION AS A FUNCTOR ON BIALGEBRAS

CHRISTIAN BROUDER AND WILLIAM SCHMITT

Abstract. The Hopf algebra of renormalization in quantum field theory is described at a general level. The products of fields at a point are assumed to form a bialgebra $\mathcal{B}$ and renormalization endows $T(T(\mathcal{B})^+)$, the double tensor algebra of $\mathcal{B}$, with the structure of a noncommutative bialgebra. When the bialgebra $\mathcal{B}$ is commutative, renormalization turns $S(S(\mathcal{B})^+)$, the double symmetric algebra of $\mathcal{B}$, into a commutative bialgebra. The usual Hopf algebra of renormalization is recovered when the elements of $S^1(\mathcal{B})$ are not renormalized, i.e., when Feynman diagrams containing one single vertex are not renormalized. When $\mathcal{B}$ is the Hopf algebra of a commutative group, a homomorphism is established between the bialgebra $S(S(\mathcal{B})^+)$ and the Faà di Bruno bialgebra of composition of series. The relation with the Connes-Moscovici Hopf algebra is given. Finally, the bialgebra $S(S(\mathcal{B})^+)$ is shown to give the same results as the standard renormalization procedure for the scalar field.

1. Introduction

The algebraic structure of quantum fields has been thoroughly studied but, until recently, their natural coalgebraic structure has not been exploited. In references [6, 7, 9] we used the coalgebraic structure of quantum fields to show that quantum groups and Hopf algebras provide an interesting tool for quantum field theory calculations. In [9], the relation between quantum groups and free scalar fields was presented at an elementary level. In [6, 7], quantum groups were employed to calculate interacting quantum fields and the coalgebra structure of quantum fields was used to derive general expressions for the time-ordered and operator products. Moreover the cohomology theory of Hopf algebras was found useful to handle time-ordered products. In the present paper, the renormalization of time-ordered products is described in detail.

In [12], Connes and Kreimer defined a Hopf algebra on Feynman diagrams that describes the renormalization of quantum field theory[1]. A little later, Gracia-Bondia and Lazzarini [23, 24] defined a Hopf algebra of Feynman diagrams related to the Epstein-Glaser renormalization. Then Pinter [35] derived the same algebra using partitions of a set of points; her work was the starting point of the present paper. The Hopf algebra of [23, 35] looks similar to the Connes-Kreimer algebra, but it is actually different because it allows for the renormalization of nonirreducible diagrams, and it works in the configuration space instead of the momentum space. This paper is devoted to a generalization of Pinter's construction to any bialgebra $\mathcal{B}$. In the first section, we consider the algebra $T(T(\mathcal{B})^+)$, where $T(\mathcal{B})^+$ is the

Date: April 10, 2007.

2000 *Mathematics Subject Classification.* 57T05, 81R99, 81T16, 81T20.

Key words and phrases. Bialgebras, Hopf algebras, Renormalization, Quantum field theory.

[1]The Hopf algebra of Connes-Kreimer belongs to a class of Hopf algebras investigated by Schmitt in 1994 [40].

nonunital subalgebra $\bigoplus_{n\geq 1} T^n(\mathcal{B})$ of the tensor algebra $T(\mathcal{B}) = \bigoplus_{n\geq 0} T^n(\mathcal{B})$. We describe how the coproduct of $\mathcal{B}$ extends freely to define bialgebra structures on $T(\mathcal{B})$ and $T(T(\mathcal{B})^+)$. These *free bialgebras* are noncommutative, and are cocommutative if and only if $\mathcal{B}$ is cocommutative. We then show that $T(T(\mathcal{B})^+)$ can be equipped with a very different coalgebra structure, making it a graded bialgebra which is neither commutative nor cocommutative, regardless of whether or not $\mathcal{B}$ is cocommutative. The abelianization of the bialgebra $T(T(\mathcal{B})^+)$ gives us the commutative bialgebra $S(T(\mathcal{B})^+)$, and $S(S(\mathcal{B})^+)$ is shown to be a subbialgebra of $S(T(\mathcal{B})^+)$. In quantum field applications, the bialgebra which is relevant to renormalization is $S(S(\mathcal{B})^+)$. When $\mathcal{B}$ is the Hopf algebra of a commutative group, we define a homomorphism from $S(S(\mathcal{B})^+)$ onto the Faà di Bruno bialgebra, which shows that $S(S(\mathcal{B})^+)$ is a kind of generalization of the algebra of formal diffeomorphisms. Hopf algebras can be obtained from $T(T(\mathcal{B})^+)$ and $S(S(\mathcal{B})^+)$ as quotients by certain biideals. We refer to these Hopf algebras as the noncommutative and commutative Pinter algebra, respectively. All of our constructions are functorial in $\mathcal{B}$ and the various mappings between them correspond to natural transformations. We describe the connection between the commutative Pinter algebra and the Connes-Moscovici algebra. Finally, we prove that our construction gives the same results as the standard renormalization procedure for scalar fields.

2. The renormalization bialgebra

In all that follows $\mathcal{B}$ is a (not necessarily unital) bialgebra over a field of characteristic zero, with product $\mu_{\mathcal{B}}$, coproduct $\delta_{\mathcal{B}}$, and counit $\varepsilon_{\mathcal{B}}$. We denote the product of two elements x, y of $\mathcal{B}$ by $x\cdot y$. We write $T(\mathcal{B})^+$ for the subalgebra $\bigoplus_{n\geq 1} T^n(\mathcal{B})$ of the tensor algebra $T(\mathcal{B})$, and denote the generators $x_1\otimes\cdots\otimes x_n$ (where $x_i \in \mathcal{B}$) of the vector space $T^n(\mathcal{B})$ by $(x_1,\dots,x_n)$; in particular, elements of $T^1(\mathcal{B}) \cong \mathcal{B}$ have the form (x), for $x \in \mathcal{B}$. We use the symbol $\circ$, rather than $\otimes$, for the product in $T(\mathcal{B})$, so that $(x_1,\dots,x_n)\circ(x_{n+1},\dots,x_{n+m}) = (x_1,\dots,x_{n+m})$. By associativity, we may consider the product $\mu_{\mathcal{B}}$ as a map $T(\mathcal{B})^+ \to \mathcal{B}$; hence $\mu_{\mathcal{B}}(x_1,\dots,x_n) = x_1\cdots x_n$, for all $(x_1,\dots,x_n) \in T^n(\mathcal{B})$. We denote the product operation in $T(T(\mathcal{B})^+)$ by juxtaposition, so that $T^k(T(\mathcal{B})^+)$ is generated by the elements $a_1a_2\cdots a_k$, where $a_i \in T(\mathcal{B})^+$. We denote the product of $a_1,\dots,a_k \in T(\mathcal{B})^+$ in $T(T(\mathcal{B})^+)$ by $\Pi_{i=1}^k a_i$, and we write $\bigotimes_{i=1}^k a_i$ for their product in $T(\mathcal{B})$.

2.1. **The free bialgebra structure on $T(T(\mathcal{B})^+)$.** The coproduct $\delta_{\mathcal{B}}$ and counit $\varepsilon_{\mathcal{B}}$ extend uniquely to a coproduct and counit on $T(\mathcal{B})$ that are compatible with the multiplication of $T(\mathcal{B})$, thus making $T(\mathcal{B})$ a bialgebra. We remark that this construction ignores completely the algebra structure of $\mathcal{B}$; the bialgebra $T(\mathcal{B})$ is in fact the free bialgebra on the underlying coalgebra of $\mathcal{B}$. Similarly, the coproduct and counit of the nonunital bialgebra $T(\mathcal{B})^+$ extend to define a free bialgebra structure on $T(T(\mathcal{B})^+)$. We denote the coproduct of both $T(\mathcal{B})$ and $T(T(\mathcal{B})^+)$ by δ. Hence, if we use the Sweedler notation $\delta_{\mathcal{B}}(x) = \sum x_{(1)}\otimes x_{(2)}$ for the coproduct of x in $\mathcal{B}$ then, for $a = (x^1,\dots,x^n) \in T^n(\mathcal{B})$ and $u = a^1\cdots a^k \in T^k(T(\mathcal{B})^+)$, we have

$$\delta(a) = \sum (x^1_{(1)},\dots,x^n_{(1)}) \otimes (x^1_{(2)},\dots,x^n_{(2)})$$

and

$$\delta(u) = \sum a^1_{(1)}\cdots a^k_{(1)} \otimes a^1_{(2)}\cdots a^k_{(2)}.$$

The counit is defined by $\epsilon(a) = \varepsilon_{\mathcal{B}}(x^1)\cdots\varepsilon_{\mathcal{B}}(x^n)$ and $\epsilon(u) = \epsilon(a^1)\cdots\epsilon(a^k)$. A similar construction was put forward by Florent Hivert [26].

For any linear map of vector spaces $f\colon V \to W$, we denote by $T(f)$ the corresponding algebra map $T(V) \to T(W)$, given by $(x_1,\dots,x_n) \mapsto (f(x_1),\dots,f(x_n))$, for all $(x_1,\dots,x_n) \in T(V)$. Note that, in particular, the map $T(\mu_{\mathcal{B}})\colon T(T(\mathcal{B})^+) \to T(\mathcal{B})$ satisfies

$$\begin{aligned} T(\mu_{\mathcal{B}})((x_1^1,\dots,x_{r_1}^1)(x_1^2,\dots,x_{r_2}^2)\cdots(x_1^k,\dots,x_{r_k}^k)) \\ &= (\mu_{\mathcal{B}}(x_1^1,\dots,x_{r_1}^1), \mu_{\mathcal{B}}(x_1^2,\dots,x_{r_2}^2),\dots,\mu_{\mathcal{B}}(x_1^k,\dots,x_{r_k}^k)) \\ &= (x_1^1\cdots x_{r_1}^1,\ x_1^2\cdots x_{r_2}^2,\dots,\ x_1^k\cdots x_{r_k}^k). \end{aligned}$$

2.2. **Grading $T(T(\mathcal{B})^+)$ by compositions.** A *composition* ρ is a (possibly empty) finite sequence of positive integers, usually referred to as the *parts* of ρ. We denote by $\ell(\rho)$ the length, that is the number of parts, of ρ, write $|\rho|$ for the sum of the parts, and say that ρ is a *composition of* n in the case that $|\rho| = n$. We denote by C_n the set of all compositions of n, and by C the set $\bigcup_{n\geq 0} C_n$ of all compositions of all nonnegative integers. For example, $\rho = (1,3,1,2)$ is a composition of 7 having length 4. The first four C_n are $C_0 = \{e\}$, where e is the empty composition, $C_1 = \{(1)\}$, $C_2 = \{(1,1),(2)\}$ and $C_3 = \{(1,1,1),(1,2),(2,1),(3)\}$. The total number of compositions of n is 2^{n-1}, the number of compositions of n of length k is $\frac{(n-1)!}{(k-1)!(n-k)!}$ and the number of compositions of n with precisely α_i occurrences of the integer i, for all i (and hence $\sum i\alpha_i = n$) is $\frac{(\alpha_1+\cdots+\alpha_n)!}{\alpha_1!\dots\alpha_n!}$.

The set C is a monoid under the operation $\circ$ of concatenation of sequences: $(r_1,\dots,r_n)\circ(r_{n+1},\dots,r_{n+m}) = (r_1,\dots,r_{n+m})$. The identity element of C is the empty composition e.

For any composition $\sigma = (s_1,\dots,s_k)$, and $1 \leq i \leq k$, let σ_i be the interval $\{s_1+\cdots+s_{i-1}+1,\cdots,s_1+\cdots+s_i\}$. The set of all compositions is partially ordered by setting $\rho \leq \sigma$ if and only if each part of σ is a sum of parts of ρ, that is, if and only if each interval σ_i is a union of ρ_j's. This order relation is called *refinement*, and will play an essential role in the definition of the bialgebra of renormalization. For compositions $\rho \leq \sigma$, with $\sigma = (s_1,\dots,s_\ell)$ and $\rho = (r_1,\dots,r_k)$, and $1 \leq i \leq \ell$, we define the *restriction* $\rho|\sigma_i$ as the composition $(r_{j_i},\dots,r_{k_i})$ of s_i, where $j_i = \min\{j :\ \rho_j \subseteq \sigma_i\}$ and $k_i = \max\{j :\ \rho_j \subseteq \sigma_i\}$. Note that we then have the factorization $\rho = (\rho|\sigma_1)\circ\cdots\circ(\rho|\sigma_k)$.

For example, if $\sigma = (4,5)$ and $\rho = (1,2,1,2,2,1)$, then $\rho = (\rho|\sigma_1)\circ(\rho|\sigma_2)$ where $(\rho|\sigma_1) = (1,2,1)$ is a composition of 4 and $(\rho|\sigma_2) = (2,2,1)$ is a composition of 5. Thus $\rho \leq \sigma$ and we say that ρ is a refinement of σ. Note that $|\rho| = |\sigma|$ and $\ell(\rho) \geq \ell(\sigma)$ if $\rho \leq \sigma$.

If $\rho \leq \sigma$, we define the *quotient* σ/ρ to be the composition of $\ell(\rho)$ given by $(t_1,\dots,t_k)$, where $t_i = \ell((\rho|\sigma_i))$, for $1 \leq i \leq k$. In our example $\sigma/\rho = (3,3)$. Note that, for $\sigma \in C_n$ with $\ell(\sigma) = k$, we have $(n)/\sigma = (k)$, $\sigma/(1,1,\dots,1) = \sigma$, and $\sigma/\sigma = (1,1,\dots,1) \in C_k$

Each of the sets C_n (as well as all of C) is partially ordered by refinement. Each C_n has unique minimal element $(1,\dots,1)$ and unique maximal element (n), and these are all the minimal and maximal elements in C. The partially ordered sets C_n are actually Boolean algebras, but we will not use this fact here.

Now we give a lemma that will be used in proving the coassociativity of the coproduct.

Lemma 1. *If $\rho \leq \tau$ in C, then the map $\sigma \mapsto \sigma/\rho$ is a bijection from the set $\{\sigma : \rho \leq \sigma \leq \tau\}$ onto the set $\{\gamma : \gamma \leq \tau/\rho\}$.*

Proof. Suppose that $\rho = (r_1, \ldots, r_k)$ and that $\gamma = (s_1, \ldots, s_\ell) \leq \tau/\rho$. Define $\bar{\gamma} \in C$ by

$$\bar{\gamma} = (r_1 + \cdots + r_{s_1},\, r_{s_1+1} + \cdots + r_{s_1+s_2},\, \ldots\, ,\, r_{k-s_\ell+1} + \cdots + r_k).$$

It is then readily verified that $\rho \leq \bar{\gamma} \leq \tau$, and that the map $\gamma \mapsto \bar{\gamma}$ is inverse to the map $\sigma \mapsto \sigma/\rho$. □

The monoid of compositions allows us to define a grading on $T(T(\mathcal{B})^+)$.

For all $n \geq 0$ and $\rho = (r_1, \ldots, r_k)$ in C_n, we let $T^\rho(\mathcal{B})$ denote the subspace of $T^k(T(\mathcal{B})^+)$ given by $T^{r_1}(\mathcal{B}) \otimes \cdots \otimes T^{r_k}(\mathcal{B})$. We then have the direct sum decomposition $T(T(\mathcal{B})^+) = \bigoplus_{\rho \in C} T^\rho(\mathcal{B})$, where $T^\rho(\mathcal{B}) \cdot T^\tau(\mathcal{B}) \subseteq T^{\rho\circ\tau}(\mathcal{B})$ for all $\rho, \tau \in C$, and $1_{T(T(\mathcal{B}))} \in T^e(\mathcal{B})$, where e denotes the empty composition; in other words, $T(T(\mathcal{B})^+)$ is a C-graded algebra.

We use this grading to define operations on $T(T(\mathcal{B})^+)$. Given $a = (x_1, \ldots, x_n) \in T^n(\mathcal{B})$ and $\rho = (r_1, \ldots, r_k) \in C$, we define $a|\rho_i \in T^{r_i}(\mathcal{B})$, for $1 \leq i \leq k$ by

$$a|\rho_i = \bigotimes_{j \in \rho_i}(x_j) = (x_{r_1+\cdots+r_{i-1}+1}, \ldots, x_{r_1+\cdots+r_i}),$$

where we take $x_j = 0$, for $j > n$, and we define the *restriction* $a|\rho \in T^\rho(\mathcal{B})$ and *contraction* $a/\rho \in T^k(\mathcal{B})$ by

$$\begin{aligned} a|\rho &= (a|\rho_1) \cdots (a|\rho_k) \\ &= (x_1, \ldots, x_{r_1})(x_{r_1+1}, \ldots, x_{r_1+r_2}) \cdots (x_{n-r_k+1}, \ldots, x_n) \end{aligned}$$

and

$$\begin{aligned} a/\rho &= T(\mu_{\mathcal{B}})(a|\rho) = (\mu_{\mathcal{B}}(a|\rho_1), \ldots, \mu_{\mathcal{B}}(a|\rho_k)) \\ &= (x_1 \cdots x_{r_1},\, x_{r_1+1} \cdots x_{r_1+r_2}, \ldots,\, x_{n-r_k+1} \cdots x_n), \end{aligned}$$

where $x_i \cdots x_j$ denotes the product of $x_i, \ldots, x_j$ in $\mathcal{B}$. Note that $a|\rho$ and a/ρ are zero if $|\rho| \neq n$. Observe also that, even though the quantity $a|\rho_i$ depends (up to a scalar multiple) on the choice of elements $x_1, \ldots, x_n \in \mathcal{B}$ representing $a = (x_1, \ldots, x_n) \in T^n(\mathcal{B})$, the quantities $a|\rho$ and a/ρ depend only on a and ρ.

More generally, for $u = a_1 \cdots a_\ell$ in $T^\sigma(\mathcal{B})$, and $\rho \in C$, we define $u|\rho \in T^\rho(\mathcal{B})$ and $u/\rho \in T^{\sigma/\rho}(\mathcal{B})$ by

$$u|\rho = a_1|(\rho|\sigma_1) \cdots a_\ell|(\rho|\sigma_\ell) \quad \text{and} \quad u/\rho = a_1/(\rho|\sigma_1) \cdots a_\ell/(\rho|\sigma_\ell).$$

Note that $u|\rho$ and u/ρ are both zero if $\rho \not\leq \sigma$.

Example 1. Suppose that $\rho = (1, 2, 1, 2, 2, 1)$, $\sigma = (3, 1, 2, 3)$ and $\tau = (4, 5)$, so that $\rho \leq \sigma \leq \tau$ in C. We then have the quotients $\sigma/\rho = (2, 1, 1, 2)$, $\tau/\rho = (3, 3)$ and $\tau/\sigma = (2, 2)$. If $u = (x_1, x_2, x_3, x_4)(y_1, y_2, y_3, y_4, y_5)$ in $T^\tau(\mathcal{B})$, then

$$\begin{aligned} u|\rho &= (x_1)(x_2, x_3)(x_4)(y_1, y_2)(y_3, y_4)(y_5) \in T^\rho(\mathcal{B}), \\ u|\sigma &= (x_1, x_2, x_3)(x_4)(y_1, y_2)(y_3, y_4, y_5) \in T^\sigma(\mathcal{B}), \\ u/\rho &= (x_1, x_2 \cdot x_3, x_4)(y_1 \cdot y_2, y_3 \cdot y_4, y_5) \in T^{\tau/\rho}(\mathcal{B}), \\ u/\sigma &= (x_1 \cdot x_2 \cdot x_3, x_4)(y_1 \cdot y_2, y_3 \cdot y_4 \cdot y_5) \in T^{\tau/\sigma}(\mathcal{B}), \\ (u|\sigma)/\rho &= (u/\rho)|(\sigma/\rho) = (x_1, x_2 \cdot x_3)(x_4)(y_1 \cdot y_2)(y_3 \cdot y_4, y_5) \in T^{\sigma/\rho}(\mathcal{B}). \end{aligned}$$

The last equality illustrates the following lemma.

Lemma 2. *For all $\rho, \sigma, \tau \in C$, and $u \in T^\tau(\mathcal{B})$, the equalities*

$$(u|\sigma)|\rho = u|\rho, \quad (u/\rho)/(\sigma/\rho) = u/\sigma, \quad \text{and} \quad (u|\sigma)/\rho = (u/\rho)|(\sigma/\rho)$$

hold in $T(T(\mathcal{B})^+)$.

Proof. First note that all the expressions above are zero unless $\rho \leq \sigma \leq \tau$ in C. Thus it suffices to prove the result for $u = a = (x_1, \ldots, x_n) \in T^n(\mathcal{B})$ and $\rho \leq \sigma$ in C_n. If $\ell(\sigma) = k$, then $a|\sigma = \prod_{i=1}^k a|\sigma_i$, and

$$(a|\sigma)|\rho = \prod_{i=1}^k (a|\sigma_i)|(\rho|\sigma_i) = \prod_{i=1}^k \prod_{\rho_j \subseteq \sigma_i} (a|\sigma_i)|\rho_j = \prod_{i=1}^k \prod_{\rho_j \subseteq \sigma_i} (a|\rho_j),$$

which is equal to $a|\rho$. For the second equation, we have

$$\begin{aligned}(a/\rho)/(\sigma/\rho) &= (\mu_{\mathcal{B}}((a/\rho)|(\sigma/\rho)_1), \ldots, \mu_{\mathcal{B}}((a/\rho)|(\sigma/\rho)_k)) \\ &= (\mu_{\mathcal{B}}(a|\sigma_1), \ldots, \mu_{\mathcal{B}}(a|\sigma_k)) \\ &= a/\sigma.\end{aligned}$$

Finally, we have

$$\begin{aligned}(a|\sigma)/\rho &= \prod_{i=1}^k (a|\sigma_i)/(\rho|\sigma_i) \\ &= \prod_{i=1}^k \bigotimes_{\rho_j \subseteq \sigma_i} (\mu_{\mathcal{B}}((a|\sigma_i)|\rho_j)) \\ &= \prod_{i=1}^k \bigotimes_{\rho_j \subseteq \sigma_i} (\mu_{\mathcal{B}}(a|\rho_j)) \\ &= \prod_{i=1}^k (a/\rho)|(\sigma/\rho)_i,\end{aligned}$$

which is equal to $(a/\rho)|(\sigma/\rho)$. □

Lemma 3. *For all $\rho \in C$ the restriction and contraction maps, given by $u \mapsto u|\rho$ and $u \mapsto u/\rho$, respectively, for homogeneous u, are coalgebra maps $T(T(\mathcal{B})^+) \to T(T(\mathcal{B})^+)$; that is, for all $\rho, \sigma \in C$ and $u \in T^\sigma(\mathcal{B})$, with free coproduct $\delta(u) = \sum u_{(1)} \otimes u_{(2)}$, the equalities*

$$\delta(u|\rho) = \sum u_{(1)}|\rho \otimes u_{(2)}|\rho, \quad \text{and}$$

$$\delta(u/\rho) = \sum u_{(1)}/\rho \otimes u_{(2)}/\rho$$

hold.

Proof. First note that both sides of each of the above equations are zero if it is not the case that $\rho \leq \sigma$ in C. Hence, by multiplicativity of δ it suffices to consider the case in which $u = a = (x^1, \ldots, x^n) \in T^n(\mathcal{B})$ and $\rho = (r_1, \ldots, r_k) \in C_n$. Then we

have

$$\begin{aligned}
\delta(a|\rho) &= \delta((x^1,\dots,x^{r_1})(x^{r_1+1},\dots,x^{r_2})\cdots(x^{n-r_k+1},\dots,x^n)) \\
&= \sum((x^1_{(1)},\dots,x^{r_1}_{(1)})(x^{r_1+1}_{(1)},\dots,x^{r_2}_{(1)})\cdots(x^{n-r_k+1}_{(1)},\dots,x^n_{(1)})) \\
&\quad \otimes((x^1_{(2)},\dots,x^{r_1}_{(2)})(x^{r_1+1}_{(2)},\dots,x^{r_2}_{(2)})\cdots(x^{n-r_k+1}_{(2)},\dots,x^n_{(2)})) \\
&= \sum(x^1_{(1)},\dots,x^n_{(1)})|\rho\otimes(x^1_{(2)},\dots,x^n_{(2)})|\rho \\
&= \sum a_{(1)}|\rho\otimes a_{(2)}|\rho,
\end{aligned}$$

and

$$\begin{aligned}
\delta(a/\rho) &= \delta(x^1\cdots x^{r_1},\dots,x^{n-r_k+1}\cdots x^n) \\
&= \sum((x^1\cdots x^{r_1})_{(1)},\dots,(x^{n-r_k+1}\cdots x^n)_{(1)}) \\
&\quad \otimes((x^1\cdots x^{r_1})_{(2)},\dots,(x^{n-r_k+1}\cdots x^n)_{(2)}) \\
&= \sum(x^1_{(1)}\cdots x^{r_1}_{(1)},\dots,x^{n-r_k+1}_{(1)}\cdots x^n_{(1)}) \\
&\quad \otimes(x^1_{(2)}\cdots x^{r_1}_{(2)},\dots,x^{n-r_k+1}_{(2)}\cdots x^n_{(2)}) \\
&= \sum(x^1_{(1)},\dots,x^n_{(1)})/\rho\otimes(x^1_{(2)},\dots,x^n_{(2)})/\rho \\
&= \sum a_{(1)}/\rho\otimes a_{(2)}/\rho.
\end{aligned}$$

□

2.3. **The renormalization coproduct and counit.** We now define the coproduct Δ on the algebra $T(T(\mathcal{B})^+)$ by

$$\Delta u = \sum_{\sigma\le\tau} u_{(1)}|\sigma\otimes u_{(2)}/\sigma, \tag{1}$$

for $u\in T^\tau(\mathcal{B})$, with free coproduct $\delta(u)=\sum u_{(1)}\otimes u_{(2)}$. The coproduct Δ is called the *renormalization coproduct* because of its role in the renormalization of quantum field theories. Note that Δ is an algebra map, and hence is determined by

$$\Delta a = \sum_{\sigma\in C_n} a_{(1)}|\sigma\otimes a_{(2)}/\sigma,$$

for all $a\in T^n(\mathcal{B})$ with $n\ge 1$. For example

$$\begin{aligned}
\Delta(x) &= \sum(x_{(1)})\otimes(x_{(2)}), \\
\Delta(x,y) &= \sum(x_{(1)})(y_{(1)})\otimes(x_{(2)},y_{(2)})+\sum(x_{(1)},y_{(1)})\otimes(x_{(2)}\cdot y_{(2)}),
\end{aligned}$$

$$\begin{aligned}
\Delta(x,y,z) &= \sum(x_{(1)})(y_{(1)})(z_{(1)})\otimes(x_{(2)},y_{(2)},z_{(2)}) \\
&\quad +\sum(x_{(1)})(y_{(1)},z_{(1)})\otimes(x_{(2)},y_{(2)}\cdot z_{(2)}) \\
&\quad +\sum(x_{(1)},y_{(1)})(z_{(1)})\otimes(x_{(2)}\cdot y_{(2)},z_{(2)}) \\
&\quad +\sum(x_{(1)},y_{(1)},z_{(1)})\otimes(x_{(2)}\cdot y_{(2)}\cdot z_{(2)}).
\end{aligned}$$

The counit ε of $T(T(\mathcal{B})^+)$ is the algebra map $T(T(\mathcal{B})^+)\to\mathbb{C}$ whose restriction to $T(\mathcal{B})^+$ is given by $\varepsilon((x))=\varepsilon_{\mathcal{B}}(x)$, for $x\in\mathcal{B}$, and $\varepsilon((x_1,\dots,x_n))=0$, for $n>1$.

Theorem 1. *The algebra $T(T(\mathcal{B})^+)$, together with the structure maps Δ and ε defined above, is a bialgebra, called the* renormalization bialgebra.

Proof. For $u \in T^\tau(\mathcal{B})$, we have

$$\begin{aligned}(\Delta \otimes \mathrm{Id})\Delta u &= \sum_{\sigma \leq \tau} \Delta(u_{(1)}|\sigma) \otimes u_{(2)}/\sigma \\ &= \sum_{\rho \leq \sigma \leq \tau} (u_{(1)}|\sigma)|\rho \otimes (u_{(2)}|\sigma)/\rho \otimes u_{(3)}/\sigma \\ &= \sum_{\rho \leq \sigma \leq \tau} u_{(1)}|\rho \otimes (u_{(2)}|\sigma)/\rho \otimes u_{(3)}/\sigma, \qquad (2)\end{aligned}$$

where the second equality is by Lemma 3, and the third by Lemma 2. On the other hand,

$$\begin{aligned}(\mathrm{Id} \otimes \Delta)\Delta u &= \sum_{\rho \leq \tau} u_{(1)}|\rho \otimes \Delta(u_{(2)}/\rho) \\ &= \sum_{\rho \leq \tau} \sum_{\gamma \leq \tau/\rho} u_{(1)}|\rho \otimes (u_{(2)}/\rho)|\gamma \otimes (u_{(3)}/\rho)/\gamma \\ &= \sum_{\rho \leq \sigma \leq \tau} u_{(1)}|\rho \otimes (u_{(2)}/\rho)|(\sigma/\rho) \otimes (u_{(3)}/\rho)/(\sigma/\rho), \qquad (3)\end{aligned}$$

where the second equality is by Lemma 3 and the third follows from Lemma 1. Expressions 2 and 3 are equal by Lemma 2, and hence Δ is coassociative.

For $a = (x^1, \ldots, x^n) \in T^n(\mathcal{B})$, with $\delta(a) = \sum a_{(1)} \otimes a_{(2)}$, we have

$$(\mathrm{Id} \otimes \varepsilon)\Delta a = \sum_{\sigma \in C_n} a_{(1)}|\sigma\, \varepsilon(a_{(2)}/\sigma).$$

Now $\varepsilon(a_{(2)}/\sigma) = 0$ unless $\sigma = (n)$; hence

$$\begin{aligned}(\mathrm{Id} \otimes \varepsilon)\Delta a &= \sum (x^1_{(1)}, \ldots, x^n_{(1)}) \varepsilon_{T(\mathcal{B})}(x^1_{(2)}, \ldots, x^n_{(2)}) \\ &= \sum (x^1_{(1)}, \ldots, x^n_{(1)}) \varepsilon_{\mathcal{B}}(x^1_{(2)}) \cdots \varepsilon_{\mathcal{B}}(x^n_{(2)}) \\ &= \sum (x^1_{(1)} \varepsilon_{\mathcal{B}}(x^1_{(2)}), \ldots, x^n_{(1)} \varepsilon_{\mathcal{B}}(x^n_{(2)})) \\ &= a.\end{aligned}$$

The proof that $(\varepsilon \otimes \mathrm{Id})\Delta a = a$ is similar. We have already observed that Δ and ϵ are algebra maps; hence $T(T(\mathcal{B})^+)$ is a bialgebra. □

We now have two coproducts on $T(T(\mathcal{B})^+)$, namely the free coproduct δ and the renormalization coproduct Δ defined by Equation 1. In order to avoid confusion in the following sections, we adopt the following alternate Sweedler notation for the new coproduct:

$$\Delta u = \sum u_{[1]} \otimes u_{[2]},$$

for all $u \in T(T(\mathcal{B})^+)$. Note that, in particular,

$$\Delta(x) = \sum (x)_{[1]} \otimes (x)_{[2]} = \sum (x_{(1)}) \otimes (x_{(2)}),$$

for all $x \in \mathcal{B}$. Whenever we simply refer to the bialgebra $T(T(\mathcal{B})^+)$, we shall mean the renormalization bialgebra; we will always state explicitly when considering $T(T(\mathcal{B})^+)$ with the free bialgebra structure.

2.4. **Recursive definition of the coproduct.** The action of $\mathcal{B}$ on itself by left multiplication extends to an action $\mathcal{B} \otimes T(\mathcal{B}) \to T(\mathcal{B})$, denoted by $x \otimes a \mapsto x \triangleright a$, in the usual manner, that is

$$x \triangleright a \;=\; (x \cdot x_1, x_2, \ldots, x_n),$$

for all $x \in \mathcal{B}$ and $a = (x_1, \ldots, x_n) \in T^n(\mathcal{B})$. This action, in turn, extends to an action of $\mathcal{B}$ on $T(T(\mathcal{B})^+)$, denoted similarly by $x \otimes u \mapsto x \triangleright u$; that is:

$$x \triangleright u \;=\; (x \triangleright a_1)a_2 \cdots a_k,$$

for all $x \in \mathcal{B}$ and $u = a_1 \cdots a_k \in T(T(\mathcal{B})^+)$.

The following proposition, together with the fact that $\Delta 1 = 1 \otimes 1$ and $\Delta(x) = \sum (x_{(1)}) \otimes (x_{(2)})$ for all $x \in \mathcal{B}$, determines Δ recursively on $T(\mathcal{B})$ and hence, by multiplicativity, determines Δ on all of $T(T(\mathcal{B})^+)$.

Proposition 1. *For all $a \in T^n(\mathcal{B})$, with $n \geq 1$, and $x \in \mathcal{B}$,*

$$\Delta((x)\circ a) \;=\; \sum (x_{(1)})a_{[1]} \otimes (x_{(2)})\circ a_{[2]} + \sum (x_{(1)})\circ a_{[1]} \otimes x_{(2)} \triangleright a_{[2]}. \tag{4}$$

Proof. We denote by C'_n the set of all compositions of n whose first part is equal to 1 and write C''_n for the set difference $C_n \backslash C'_n$. Note that the map $\rho \mapsto (1)\circ\rho$ is a bijection from C_n onto C'_{n+1}. If $\rho = (r_1, r_2, \ldots, r_k)$ we define $\rho^+ = (r_1 + 1, r_2, \ldots, r_k)$ and the map $\rho \mapsto \rho^+$ is a bijection from C_n onto C''_{n+1}. From the definition of $a|\rho$ and a/ρ it can be checked that $\big((x)\circ a\big)/\big((1)\circ\rho\big) = (x)\circ(a/\rho)$, $\big((x)\circ a\big)|\big((1)\circ\rho\big) = (x)(a|\rho)$, $\big((x)\circ a\big)/\rho^+ = (x)\triangleright(a|\rho)$ and $\big((x)\circ a\big)|\rho^+ = (x)\circ(a|\rho)$, where in the last identity we extend the product of $T(\mathcal{B})$ by $(x) \circ u = \big((x) \circ a_1\big)a_2 \ldots a_k$ if $u = a_1 a_2 \ldots a_k$.

We then have

$$\begin{aligned}
\Delta((x)\circ a) \;&=\; \sum_{\rho \in C'_{n+1}} ((x)\circ a)_{(1)}|\rho \otimes ((x)\circ a)_{(2)}/\rho \\
&\quad + \sum_{\rho \in C''_{n+1}} ((x)\circ a)_{(1)}|\rho \otimes ((x)\circ a)_{(2)}/\rho \\
&=\; \sum_{\sigma \in C_n} (x_{(1)})(a_{(1)}|\sigma) \otimes (x_{(2)})\circ(a_{(2)}/\sigma) \\
&\quad + \sum_{\tau \in C_n} (x_{(1)})\circ(a_{(1)}|\tau) \otimes (x_{(2)}) \triangleright (a_{(2)}/\tau),
\end{aligned}$$

which is precisely the right-hand side of Equation 4. □

The recursive definition of Δ was used in [8] to show that $T(T(\mathcal{B})^+)$ is isomorphic to the noncommutative Hopf algebra of formal diffeomorphisms in the case that $\mathcal{B}$ is the trivial algebra.

We may formulate Equation 4 as follows: Corresponding to an element x of $\mathcal{B}$ there are three linear operators on $T(T(\mathcal{B})^+)$:

$$\begin{aligned}
A_x(u) \;&=\; x \triangleright u \\
B_x(u) \;&=\; ((x) \circ a_1)a_2 \cdots a_k \qquad (\text{where } u = a_1 \cdots a_k) \\
C_x(u) \;&=\; (x)u
\end{aligned}$$

induced by left multiplication in $\mathcal{B}$, $T(\mathcal{B})$, and $T(T(\mathcal{B})^+)$, respectively. With this notation, Equation 4 takes the form

$$\Delta(B_x(a)) \;=\; \sum (C_{x_{(1)}} \otimes B_{x_{(2)}} + B_{x_{(1)}} \otimes A_{x_{(2)}})\Delta a. \tag{5}$$

As a third formulation, let A, B and C be the mappings from $\mathcal{B}$ to the set of linear operators on $T(T(\mathcal{B})^+)$ respectively given by $x \mapsto A_x$, $x \mapsto B_x$, and $x \mapsto C_x$; then Equation 4 takes the form

$$\Delta(B_x(a)) \;=\; (B \otimes A + C \otimes B)(\delta(x))(\Delta a).$$

We also have

$$\begin{aligned} \Delta(A_x(a)) \;&=\; (A \otimes A)(\delta(x))(\Delta a), \\ \Delta(C_x(a)) \;&=\; (C \otimes C)(\delta(x))(\Delta a). \end{aligned}$$

Finally, we give a more explicit expression for the coproduct. If $a = (x_1, \ldots, x_n)$, we have

$$\Delta a \;=\; \sum_u a^1_{(1)} \cdots a^{\ell(u)}_{(1)} \otimes (\prod a^1_{(2)}, \ldots, \prod a^{\ell(u)}_{(2)}), \tag{6}$$

where the product $a^1_{(1)} \cdots a^{\ell(u)}_{(1)}$ is in $T(T(\mathcal{B})^+)$. In this formula, which is a simple rewriting of Equation 1, u runs over the compositions of a. By a composition of a, we mean an element u of $T(T(\mathcal{B})^+)$ such that $u = (a|\rho)$ for some $\rho \in C_n$. If the length of ρ is k, we can write $u = a^1 \ldots a^k$ where $a^i \in T(\mathcal{B})$ are called the blocks of u. Finally the length of u is $\ell(u) = \ell(\rho) = k$. To complete the definition of Equation 6, we still have to define $a^i_{(1)}$ and $\prod a^i_{(2)}$. If $a^i = (y^1, \ldots, y^m)$ is a block, then $a^i_{(1)} = (y^1_{(1)}, \ldots, y^m_{(1)}) \in T(\mathcal{B})^+$ and $\prod a^i_{(2)} = \mu_{\mathcal{B}}(y^1_{(2)}, \ldots, y^m_{(2)}) \in \mathcal{B}$.

2.5. **Functoriality.** Given vector spaces V, W, and a linear map $f\colon V \to W$, we denote by $\hat{f}$ the algebra map $T(T(f))\colon T(T(V)^+) \to T(T(W)^+)$, determined by $\hat{f}(a) = T(f)(a) = (f(x^1), \ldots, f(x^n))$ for all $a = (x^1, \ldots, x^n) \in T^n(V)$, and $\hat{f}(u) = \hat{f}(a_1) \cdots \hat{f}(a_k)$, for all $u = a_1 \cdots a_k \in T(T(V)^+)$, where $a_1, \ldots, a_k \in T(V)^+$. The following proposition shows that the renormalization construction on bialgebras is functorial.

Proposition 2. *If $\mathcal{B}$ and $\mathcal{C}$ are bialgebras and $f\colon \mathcal{B} \to \mathcal{C}$ is a bialgebra map, then $\hat{f}\colon T(T(\mathcal{B})^+) \to T(T(\mathcal{C})^+)$ is a bialgebra map.*

Proof. Given $a \in T^n(\mathcal{B})$, and a composition $\rho \in C_n$, it follows directly from the definition of $\hat{f}$ that $\hat{f}(a|\rho) = \hat{f}(a)|\rho$, and using the fact that f preserves products, it follows that $\hat{f}(a/\rho) = \hat{f}(a)/\rho$. By multiplicativity of $\hat{f}$ we thus have

$$\hat{f}(u|\sigma) \;=\; \hat{f}(u)|\sigma \qquad \text{and} \qquad \hat{f}(u/\sigma) \;=\; \hat{f}(u)/\sigma,$$

for all homogeneous $u \in T(T(\mathcal{B})^+)$ and all compositions $\sigma \in C$. Furthermore, it is immediate from the fact $f\colon \mathcal{B} \to \mathcal{C}$ preserves coproducts that $\hat{f}\colon T(T(\mathcal{B})^+) \to T(T(\mathcal{C})^+)$ preserves free coproducts. Thus for all compositions τ, and $u \in T^\tau(\mathcal{B})$,

with free coproduct $\delta(u) = \sum u_{(1)} \otimes u_{(2)}$, we have

$$\begin{aligned}\Delta\hat{f}(u) &= \sum_{\sigma\leq\tau} f(u_{(1)})|\sigma \otimes f(u_{(2)})/\sigma,\\ &= \sum_{\sigma\leq\tau} f(u_{(1)}|\sigma) \otimes f(u_{(2)}/\sigma),\\ &= (\hat{f}\otimes\hat{f})\Delta(u),\end{aligned}$$

and so $\hat{f}$ preserves the renormalization coproduct. □

2.6. **Grading.** We now assume that $\mathcal{B}$ is a graded bialgebra; this entails no loss of generality because we can always consider that all elements of $\mathcal{B}$ have degree 0. The grading of $\mathcal{B}$ will be used to define a grading on the bialgebra $T(T(\mathcal{B})^+)$. We denote by $|x|$ the degree of a homogeneous element x of $\mathcal{B}$, and by $\deg(a)$ the degree (to be defined) of homogeneous a in $T(T(\mathcal{B})^+)$. We first discuss the grading of elements of $T(\mathcal{B})$. The degree of 1 is zero, the degree of $(x) \in T^1(\mathcal{B})$ is equal to the degree of x in $\mathcal{B}$, that is, $\deg\big((x)\big) = |x|$. More generally, the degree of $(x_1,\dots,x_n) \in T^n(\mathcal{B})$ is

$$\deg\big((x_1,\dots,x_n)\big) = |x_1| + \cdots + |x_n| + n - 1.$$

Finally, if $a_1,\dots,a_k$ are homogeneous elements of $T(\mathcal{B})^+$, the degree of their product in $T(T(\mathcal{B})^+)$ is defined by

$$\deg(a_1 \dots a_k) = \deg(a_1) + \cdots + \deg(a_k).$$

Proposition 3. *The renormalization bialgebra $T(T(\mathcal{B})^+)$, with degree defined as above, is a graded bialgebra.*

Proof. By definition, the degree is compatible with the multiplication of $T(T(\mathcal{B})^+)$. The fact that it is also compatible with the renormalization coproduct of $T(T(\mathcal{B})^+)$, follows directly from Formula 6. □

For dealing with fermions, we must use a $\mathbb{Z}_2$-graded algebra $\mathcal{B}$. In this case, we extend the grading of $\mathcal{B}$ to a $\mathbb{Z}_2$-grading of $T(T(\mathcal{B})^+)$ as follows: for $a = (x^1,\dots,x^n) \in T^n(\mathcal{B})$, we set $|a| = |x^1|+\cdots+|x^n|$, and for $u = a_1\cdots a_k \in T(T(\mathcal{B})^+)$, we set $|u| = |a_1| + \cdots + |a_k|$. The coproduct is determined by

$$\Delta a = \sum_{\rho\in C_n} \operatorname{sgn}(a_{(1)}, a_{(2)})\, a_{(1)}|\ \rho \otimes a_{(2)}/\rho,$$

for $a = (x^1,\dots,x^n) \in T^n(\mathcal{B})$, where $\delta(a) = \sum a_{(1)} \otimes a_{(2)}$ is the free coproduct and

$$\operatorname{sgn}(a_{(1)}, a_{(2)}) = (-1)^{\sum_{k=2}^{n}\sum_{l=1}^{k-1}|x^k_{(1)}||x^l_{(2)}|}.$$

is the usual Koszul sign factor.

3. The commutative renormalization bialgebra

When the bialgebra $\mathcal{B}$ is commutative, it is possible to work with the symmetric algebra $S(S(\mathcal{B})^+)$ instead of the tensor algebra $T(T(\mathcal{B})^+)$. We construct $S(S(\mathcal{B})^+)$ as a subbialgebra of the quotient bialgebra $S(T(\mathcal{B})^+)$ of $T(T(\mathcal{B})^+)$. We denote by Σ_n the set of all permutations of $\{1,\dots,n\}$ and, for $a = (x_1,\dots,x_n) \in T^n(\mathcal{B})$ and $\sigma \in \Sigma_n$, we write a^σ for $(x_{\sigma(1)},\dots,x_{\sigma(n)})$. Let $\alpha = \alpha_{\mathcal{B}}\colon T(\mathcal{B}) \to T(\mathcal{B})$ be the symmetrizing operator given by $a \mapsto \sum a^\sigma$, for all $a \in T^n(\mathcal{B})$, where the sum is

over all $\sigma \in \Sigma_n$. We identify $S(\mathcal{B})^+$, as a vector space, with the image $\alpha(T(\mathcal{B})^+)$ in $T(\mathcal{B})$. We write $\{a\}$ for $\alpha(a)$ and denote the product in $S(\mathcal{B})^+$ by $\vee$, so that

$$\{x_1, \ldots, x_n\} = \sum_{\sigma \in \Sigma_n} (x_{\sigma(1)}, \ldots, x_{\sigma(n)}), \tag{7}$$

and $\{x_1, \ldots, x_n\} \vee \{x_{n+1}, \ldots, x_{n+m}\} = \{x_1, \ldots, x_{n+m}\}$, for all $x_1, \ldots, x_{n+m} \in \mathcal{B}$. Note that $S(\mathcal{B})^+$, with product $\vee$ is not a subalgebra of $T(\mathcal{B})^+$.

We define $S(T(\mathcal{B})^+)$ as the quotient algebra $T(T(\mathcal{B})^+)/I$, where I is the ideal $\{u(ab-ba)v \,|\, u, v \in T(T(\mathcal{B})^+)$ and $a, b \in T(\mathcal{B})^+\}$. The symmetric algebra $S(S(\mathcal{B})^+)$ is the image of the subspace $T(S(\mathcal{B})^+) \subseteq T(T(\mathcal{B})^+)$ under the canonical projection $T(T(\mathcal{B})^+) \to S(T(\mathcal{B})^+)$ or, equivalently, the image of the map $S(\alpha)\colon S(T(\mathcal{B})^+) \to S(T(\mathcal{B})^+)$, determined by $S(\alpha)(a_1 \cdots a_k) = \{a_1\} \cdots \{a_n\}$, for all $a_1, \ldots, a_k \in T(\mathcal{B})^+$.

The following lemma shows that the symmetric algebra $S(T(\mathcal{B})^+)$ inherits the renormalization coproduct.

Lemma 4. *The symmetric algebra $S(T(\mathcal{B})^+)$ is a quotient bialgebra of the renormalization bialgebra $T(T(\mathcal{B})^+)$.*

Proof. The bialgebra $S(T(\mathcal{B})^+)$ is obtained from the bialgebra $T(T(\mathcal{B})^+)$ by the standard quotient method (see, e.g., [30], p.56). We define the ideal $I = \{u(ab - ba)v \,|\, u, v \in T(T(\mathcal{B})^+), a, b \in T(\mathcal{B})^+\}$, and note that

$$\begin{aligned}
\Delta u(ab-ba)v &= \sum u_{[1]}a_{[1]}b_{[1]}v_{[1]} \otimes u_{[2]}a_{[2]}b_{[2]}v_{[2]} \\
&\quad - \sum u_{[1]}b_{[1]}a_{[1]}v_{[1]} \otimes u_{[2]}b_{[2]}a_{[2]}v_{[2]} \\
&= \sum u_{[1]}(a_{[1]}b_{[1]} - b_{[1]}a_{[1]})v_{[1]} \otimes u_{[2]}a_{[2]}b_{[2]}v_{[2]} \\
&\quad + \sum u_{[1]}b_{[1]}a_{[1]}v_{[1]} \otimes u_{[2]}(a_{[2]}b_{[2]} - b_{[2]}a_{[2]})v_{[2]}.
\end{aligned}$$

Thus $\Delta I \subset I \otimes T(T(\mathcal{B})^+) + T(T(\mathcal{B})^+) \otimes I$. Moreover, $\varepsilon(I) = 0$ because $\varepsilon(ab - ba) = 0$. Therefore I is a coideal. Since I is also an ideal, the quotient $S(T(\mathcal{B})^+) = T(T(\mathcal{B})^+)/I$ is a bialgebra, which is commutative [30]. □

In order to describe the commutative renormalization bialgebra $S(S(\mathcal{B})^+)$, we first establish some notation involving partitions of sets. A *partition* of a set S is a set π of nonempty, pairwise disjoint, subsets of S, called the *blocks* of π, having union equal to S. We denote by Π_n the set of all partitions of $\{1, \ldots, n\}$. Given $a = (x_1, \ldots, x_n) \in T^n(\mathcal{B})$ and a subset $B = \{i_1, \ldots, i_j\}$ of $\{1, \ldots, n\}$, we define $\{a|B\} \in S^j(\mathcal{B})$ by

$$\{a|B\} = \{x_{i_1}, \ldots, x_{i_k}\} = \bigvee_{i \in B} \{x_i\},$$

and, for $\pi = \{B_1, \ldots, B_k\} \in \Pi_n$, we define $a|\pi \in S(S(\mathcal{B})^+)$ by

$$a|\pi = \{a|B_1\} \cdots \{a|B_k\} = \prod_{B \in \pi} \{a|B\}.$$

If $\mathcal{B}$ is commutative, we regard the product $\mu_{\mathcal{B}}$ as a map $S(\mathcal{B})^+ \to B$, and in this case we define $a/\pi \in S(\mathcal{B})^+$ by

$$a/\pi = S(\mu_{\mathcal{B}})(a|\pi) = \{\mu_{\mathcal{B}}\{a|B_1\}, \ldots, \mu_{\mathcal{B}}\{a|B_k\}\}.$$

Theorem 2. *If $\mathcal{B}$ is a commutative bialgebra then the symmetric algebra $S(S(\mathcal{B})^+)$ is a subbialgebra of $S(T(\mathcal{B})^+)$. The coproduct of $S(T(\mathcal{B})^+)$, restricted to $S(S(\mathcal{B})^+)$, is determined by the formula*

$$\Delta\{a\} = \sum_{\pi\in\Pi_n} a_{(1)}|\pi \otimes a_{(2)}/\pi, \tag{8}$$

for all $a \in T^n(\mathcal{B})$. We refer to $S(S(\mathcal{B})^+)$ as the commutative renormalization bialgebra.

We express the coproduct (8) analogously to the formula (6) for the coproduct of $T(T(\mathcal{B})^+)$ as follows:

$$\Delta b = \sum_{\pi\in\Pi_n} b^1_{(1)}\cdots b^k_{(1)} \otimes \{\prod b^1_{(2)}, \ldots, \prod b^k_{(2)}\}. \tag{9}$$

Here, $b = \{a\} \in S^n(\mathcal{B})$, where $a = (x_1, \ldots, x_n) \in T^n(\mathcal{B})$ and, for each partition $\pi \in \Pi_n$, we have $b^i_{(1)} = \{a_{(1)}|B_i\}$, and $\prod b^i_{(2)} = \mu_{\mathcal{B}}(\{a_{(2)}|B_i\})$, for $1 \le i \le k$, where $\{B_1, \ldots, B_k\}$ is the set of blocks of π.

Examples:

$$\begin{aligned}
\Delta\{x\} &= \sum \{x_{(1)}\} \otimes \{x_{(2)}\} \\
\Delta\{x,y\} &= \sum \{x_{(1)}\}\{y_{(1)}\} \otimes \{x_{(2)}, y_{(2)}\} + \sum \{x_{(1)}, y_{(1)}\} \otimes \{x_{(2)} \cdot y_{(2)}\} \\
\Delta\{x,y,z\} &= \sum \{x_{(1)}\}\{y_{(1)}\}\{z_{(1)}\} \otimes \{x_{(2)}, y_{(2)}, z_{(2)}\} \\
&\quad + \sum \{x_{(1)}\}\{y_{(1)}, z_{(1)}\} \otimes \{x_{(2)}, y_{(2)} \cdot z_{(2)}\} \\
&\quad + \sum \{y_{(1)}\}\{x_{(1)}, z_{(1)}\} \otimes \{y_{(2)}, x_{(2)} \cdot z_{(2)}\} \\
&\quad + \sum \{z_{(1)}\}\{x_{(1)}, y_{(1)}\} \otimes \{z_{(2)}, x_{(2)} \cdot y_{(2)}\} \\
&\quad + \sum \{x_{(1)}, y_{(1)}, z_{(1)}\} \otimes \{x_{(2)} \cdot y_{(2)} \cdot z_{(2)}\}. \\
\Delta\{x,y,y\} &= \sum \{x_{(1)}\}\{y_{(1)}\}\{y_{(1)}\} \otimes \{x_{(2)}, y_{(2)}, y_{(2)}\} \\
&\quad + \sum \{x_{(1)}\}\{y_{(1)}, y_{(1)}\} \otimes \{x_{(2)}, y_{(2)} \cdot y_{(2)}\} \\
&\quad + 2\sum \{y_{(1)}\}\{x_{(1)}, y_{(1)}\} \otimes \{y_{(2)}, x_{(2)} \cdot y_{(2)}\} \\
&\quad + \sum \{x_{(1)}, y_{(1)}, y_{(1)}\} \otimes \{x_{(2)} \cdot y_{(2)} \cdot y_{(2)}\}.
\end{aligned}$$

Proof of Theorem 2. We first note that, since the sum on the right-hand side of Equation 8 is over all partitions π of $\{1, \ldots, n\}$, the result is independent of the choice of $a \in T^n(\mathcal{B})$ representing $\{a\}$. The fact that $S(S(\mathcal{B})^+)$ is a subbialgebra of $S(T(\mathcal{B})^+)$ will follow once we establish Equation 8, since the right-hand side belongs to $S(S(\mathcal{B})^+) \otimes S(\mathcal{B})^+ \subseteq S(S(\mathcal{B})^+) \otimes S(S(\mathcal{B})^+)$.

The proof depends on a basic bijection relating compositions, permutations and partitions. By a *totally ordered* partition, we shall mean a partition π in some Π_n each of whose blocks is equipped with a linear ordering, and which is linearly ordered itself. Given a pair (ρ, σ), where $\rho = (r_1, \ldots, r_k)$ is a composition of n and σ is a permutation of $\{1, \ldots, n\}$, we obtain a partition $\pi = \{B_1, \ldots, B_k\}$ of $\{1, \ldots, n\}$ by setting $B_i = \sigma(\rho_i) = \{\sigma(r_1 + \cdots + r_{i-1} + 1), \ldots, \sigma(r_1 + \cdots + r_i)\}$, for $1 \le i \le k$. The set π is ordered by $B_1 < \cdots < B_k$, and each B_i has the linear ordering inherited

from the order $\sigma(1) < \cdots < \sigma(n)$ of $\{1, \ldots, n\}$. The correspondence $(\rho, \sigma) \mapsto \pi$ thus defines a bijection from the cartesian product $C_n \times \Sigma_n$ onto the set of totally ordered partitions of $\{1, \ldots, n\}$. The inverse bijection maps a totally ordered partition $\{B_1, \ldots, B_k\}$ of $\{1, \ldots, n\}$ to the pair (ρ, σ), where $\rho = (|B_1|, \ldots, |B_k|)$, and $\sigma(i)$ is the i*th* element of the concatenation of the linearly ordered sets $B_1, B_2, \ldots, B_k$, for $1 \leq i \leq n$.

Now suppose that $a = (x_1, \ldots, x_n) \in T^n(\mathcal{B})$. By definition of $\{a\}$ and the coproduct formula (1), we have

$$\Delta\{a\} = \sum_{\rho \in C_n} \sum_{\sigma \in \Sigma_n} a^\sigma_{(1)} | \rho \otimes a^\sigma_{(2)} / \rho,$$

where it is understood that, for ρ a composition of length k, the expression $a^\sigma_{(1)}|\rho = (a^\sigma_{(1)}|\rho_1) \cdots (a^\sigma_{(1)}|\rho_k)$ is the product in $S(T(\mathcal{B})^+)$. Using the commutativity of this product on the left side of the tensor product, the commutativity of $\mu_\mathcal{B}$ on the right side, and the above bijection, we thus have

$$\Delta\{a\} = \sum \{a_{(1)}|B_1\} \cdots \{a_{(1)}|B_k\} \otimes \{a_{(2)}/\pi\},$$

where the sum is over all $\pi = \{B_1, \ldots, B_k\} \in \Pi_n$; in other words

$$\Delta\{a\} = \sum_{\pi \in \Pi_n} a_{(1)} | \pi \otimes a_{(2)} / \pi.$$

□

It is also possible to identify $S(S(\mathcal{B})^+)$ as a subspace of $T(T(\mathcal{B})^+)$; that is, as the image of $T(T(\mathcal{B})^+)$ under the composition $\alpha_{T(\mathcal{B})} T(\alpha_\mathcal{B})\colon T(T(\mathcal{B})) \to T(T(\mathcal{B}))$, which maps $u = a_1 \cdots a_k \in T(T(\mathcal{B})^+)$ to $\{\{a_1\}, \ldots, \{a_k\}\} \in S(S(\mathcal{B})^+)$. The proof of Theorem 2 shows that, under this identification, the commutative renormalization bialgebra $S(S(\mathcal{B})^+)$ is in fact a subcoalgebra of the noncommutative renormalization bialgebra $T(T(\mathcal{B})^+)$.

4. The Faà di Bruno bialgebra

When $\mathcal{B}$ is the Hopf algebra of a commutative group, there is a homomorphism from the bialgebra $S(S(\mathcal{B})^+)$ to the Faà di Bruno algebra. In 1855, Francesco Faà di Bruno (who was beatified in 1988), derived the general formula for the nth derivative of the composition of two functions $f(g(x))$ [15]. In 1974, Peter Doubilet defined a bialgebra arising from the partitions of a set [16]. This bialgebra was called the Faà di Bruno bialgebra by Joni and Rota [29], because it is closely related to the Faà di Bruno formula. This bialgebra was further investigated by Schmitt in [39, 40], by Schmitt and Haiman in [25], and more recently by Figueroa and Gracia-Bondia in [22].

4.1. **Definition.** As an algebra, the Faà di Bruno bialgebra $\mathcal{F}$ is the polynomial algebra generated by u_n for $n \geq 1$. The coproduct of $\mathcal{F}$ is determined by

$$\delta u_n = \sum_{\pi \in \Pi_n} u_\pi \otimes u_{\ell(\pi)}, \tag{10}$$

where u_π denotes the product $\prod_{B \in \pi} u_{|B|}$, and $\ell(\pi)$ the number of blocks of π, for all partitions π. If $\pi \in \Pi_n$ has precisely α_i blocks of size i, for all i, then

$u_\pi = u_1^{\alpha_1} \cdots u_n^{\alpha_n}$. Since the number of such partitions is given by

$$\frac{n!}{\alpha_1!\alpha_2! \cdots \alpha_n!(1!)^{\alpha_1}(2!)^{\alpha_2} \cdots (n!)^{\alpha_n}}$$

(see, e.g., [1]), it follows that the coproduct of $\mathcal{F}$ may also be expressed as

$$\Delta u_n = \sum_{k=1}^{n} \sum_{\alpha} \frac{n!(u_1)^{\alpha_1}(u_2)^{\alpha_2} \cdots (u_n)^{\alpha_n}}{\alpha_1!\alpha_2! \cdots \alpha_n!(1!)^{\alpha_1}(2!)^{\alpha_2} \cdots (n!)^{\alpha_n}} \otimes u_k, \tag{11}$$

where the inner sum is over the n-tuples of nonnegative integers $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ such that $\alpha_1 + 2\alpha_2 + \cdots + n\alpha_n = n$ and $\alpha_1 + \alpha_2 + \cdots + \alpha_n = k$. For example,

$$\begin{aligned}
\Delta u_1 &= u_1 \otimes u_1, \\
\Delta u_2 &= u_2 \otimes u_1 + u_1^2 \otimes u_2, \\
\Delta u_3 &= u_3 \otimes u_1 + 3u_1u_2 \otimes u_2 + u_1^3 \otimes u_3, \\
\Delta u_4 &= u_4 \otimes u_1 + 4u_1u_3 \otimes u_2 + 3u_2^2 \otimes u_2 + 6u_1^2u_2 \otimes u_3 + u_1^4 \otimes u_4.
\end{aligned}$$

Since $\mathcal{F}$ is a bialgebra, the set $\mathcal{M}$ of algebra maps from $\mathcal{F}$ to the scalar field is a multiplicatively closed subset of the dual algebra $\mathcal{F}^*$. Using Sweedler notation $\delta u = \sum u_{(1)} \otimes u_{(2)}$ for the coproduct of $\mathcal{F}$, we have that the (convolution) product of $f, g \in \mathcal{M}$ is determined by $(f \star g)(u_n) = \sum f(u_{n\,(1)})g(u_{n\,(2)})$, for all n. To each element $f \in \mathcal{M}$, we associate the exponential series

$$f(x) = \sum_{n=1}^{\infty} f(u_n)\frac{x^n}{n!}.$$

It then follows directly from Equation 11 that, for all $f, g \in \mathcal{M}$, the coefficient of $x^n/n!$ in the composition $f(g(x))$ is equal to $(g \star f)(u_n)$, and thus $f(g(x)) = (g \star f)(x)$. Hence the monoid $\mathcal{M}$ is antiisomorphic to the monoid of all exponential formal power series having zero constant term, under the operation of composition.

For instance, the first few coefficients of $f(g(x))$ are

$$\begin{aligned}
(g \star f)(u_1) &= g_1f_1, \\
(g \star f)(u_2) &= g_2f_1 + g_1^2f_2, \\
(g \star f)(u_3) &= g_3f_1 + 3g_1g_2f_2 + g_1^3f_3, \\
(g \star f)(u_4) &= g_4f_1 + 4g_1g_3f_2 + 3g_2^2f_2 + 6g_1^2g_2f_3 + g_1^4f_4,
\end{aligned}$$

where we have written f_n and g_n for $f(u_n)$ and $g(u_n)$.

Following Connes and Moscovici [13], it is possible to introduce a new (noncommutative) element X in the algebra, such that $[X, u_n] = u_{n+1}$, and to generate the Faà di Bruno coproduct from the relations

$$\begin{aligned}
\Delta u_1 &= u_1 \otimes u_1, \\
\Delta X &= X \otimes 1 + u_1 \otimes X.
\end{aligned}$$

4.2. **Homomorphism.** Here, we take the bialgebra $\mathcal{B}$ to be a commutative group Hopf algebra. If G is a commutative group, the commutative algebra $\mathcal{B}$ is the vector space generated by the elements of G, with product induced by the product in G.

The coproduct is defined by $\delta_{\mathcal{B}} x = x \otimes x$ for all elements $x \in G$. Formula 9 for the coproduct gives

$$\Delta b = \sum_{\pi \in \Pi_n} b^1 \cdots b^k \otimes \{\prod b^1, \ldots, \prod b^k\}. \tag{12}$$

for $b = \{x_1, \ldots, x_n\}$ with $x_i \in G$.

The homomorphism φ from $S(S(\mathcal{B})^+)$ to the Faà di Bruno bialgebra is given by: $\varphi(1) = 1$ and $\varphi(a) = u_n$ for any $a \in S^n(\mathcal{B})$ with $n > 0$. It is clear that φ is an algebra map; the fact that it respects coproducts can be established directly by comparing Equations 12 and 10.

5. The Pinter Hopf algebra

In this section, we complete Pinter's construction, building the noncommutative and commutative Hopf algebras that can be obtained from the noncommutative and commutative renormalization bialgebras, respectively. These algebras will be called the commutative and noncommutative Pinter Hopf algebras.

5.1. **The noncommutative case.** The renormalization bialgebra $T(T(\mathcal{B})^+)$ can be turned into a Hopf algebra by quotienting by an ideal. The subspace $I = \{(x) - \varepsilon_{\mathcal{B}}(x)1, x \in \mathcal{B}\}$ of $T(T(\mathcal{B})^+)$ is a coideal because $\varepsilon(I) = 0$ and

$$\begin{aligned}
\Delta\big((x) - \varepsilon_{\mathcal{B}}(x)1\big) &= \sum (x_{(1)}) \otimes (x_{(2)}) - \varepsilon_{\mathcal{B}}(x)1 \otimes 1 \\
&= \sum (x_{(1)}) \otimes (x_{(2)}) - \sum \varepsilon_{\mathcal{B}}(x_{(1)})\varepsilon_{\mathcal{B}}(x_{(2)})1 \otimes 1 \\
&= \sum \big((x_{(1)}) - \varepsilon_{\mathcal{B}}(x_{(1)})1\big) \otimes (x_{(2)}) \\
&\quad + \sum \varepsilon_{\mathcal{B}}(x_{(1)})1 \otimes \big((x_{(2)}) - \varepsilon_{\mathcal{B}}(x_{(2)})1\big).
\end{aligned}$$

Therefore, the subspace J of elements of the form uav, where $u, v \in T(T(\mathcal{B})^+)$ and $a \in I$, is an ideal and a coideal and $T(T(\mathcal{B})^+)/J$ is a bialgebra [14]. The action of the quotient is to replace all the (x) by $\varepsilon(x)1$. For example, we have $\Delta(x, y) = 1 \otimes (x, y) + (x, y) \otimes 1$, and

$$\begin{aligned}
\Delta(x, y, z) &= 1 \otimes (x, y, z) + \sum (y_{(1)}, z_{(1)}) \otimes (x, y_{(2)} \cdot z_{(2)}) \\
&\quad + \sum (x_{(1)}, y_{(1)}) \otimes (x_{(2)} \cdot y_{(2)}, z) + (x, y, z) \otimes 1.
\end{aligned}$$

More generally, if $a = (x_1, \ldots, x_n)$, then

$$\Delta a = a \otimes 1 + 1 \otimes a + \sum{}' a_{(1)} \otimes a_{(2)},$$

where Σ' involves only elements $a_{(1)}$ and $a_{(2)}$ of degrees strictly smaller than the degree of a. Hence, $T(T(\mathcal{B})^+)/J$ is a connected Hopf algebra and the antipode can be defined as in [39].

5.2. **The commutative case.** The same construction can be carried out with $S(S(\mathcal{B})^+)/J'$, where J' is the subspace of elements of the form uav, where $u, v \in$

$S(S(\mathcal{B})^+)$ and $a \in \{\{x\} - \varepsilon_{\mathcal{B}}(x)1, x \in \mathcal{B}\}$. This gives us

$$\begin{aligned}
\Delta\{x,y\} &= 1 \otimes \{x,y\} + \{x,y\} \otimes 1,\\
\Delta\{x,y,z\} &= 1 \otimes \{x,y,z\} + \sum \{y_{(1)}, z_{(1)}\} \otimes \{x, y_{(2)} \cdot z_{(2)}\}\\
&\quad + \sum \{x_{(1)}, z_{(1)}\} \otimes \{y, x_{(2)} \cdot z_{(2)}\}\\
&\quad + \sum \{x_{(1)}, y_{(1)}\} \otimes \{z, x_{(2)} \cdot y_{(2)}\} + \{x,y,z\} \otimes 1.
\end{aligned}$$

5.3. **The Connes-Moscovici Hopf algebra.** If we take the same quotient of the Faà di Bruno bialgebra (i.e. by letting $u_1 = 1$), we obtain a Hopf algebra, that we call the Faà di Bruno Hopf algebra. In the course of a proof of index theorems in noncommutative geometry, Connes and Moscovici defined a noncommutative Hopf algebra [13]. They noticed that the commutative part of this Hopf algebra is related to the algebra of diffeomorphisms as follows: If $\phi(x) = x + \sum_{n=2}^{\infty} u_n x^n/n!$, they define δ_n for $n > 0$ by

$$\log \phi'(x) = \sum_{n=1}^{\infty} \delta_n \frac{x^n}{n!}.$$

To calculate δ_n as a function of u_k, we use the fact that $\phi'(x) = 1 + \sum_{n=1}^{\infty} u_{n+1} x^n/n!$ and $\log(1+z) = \sum_{k=1}^{\infty} (-1)^{k-1}(k-1)! z^k/k!$. Since the Faà di Bruno formula describes the composition of series, we can use it to write immediately

$$\delta_n = \sum_{k=1}^{n} (-1)^{k-1}(k-1)! \sum_{\alpha} \frac{n!(u_2)^{\alpha_1}(u_3)^{\alpha_2}\cdots(u_{n+1})^{\alpha_n}}{\alpha_1!\alpha_2!\cdots\alpha_n!(1!)^{\alpha_1}(2!)^{\alpha_2}\cdots(n!)^{\alpha_n}},$$

where the sum is over the n-tuples of nonnegative integers $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ such that $\alpha_1 + 2\alpha_2 + \cdots + n\alpha_n = n$ and $\alpha_1 + \alpha_2 + \cdots + \alpha_n = k$.

For example, $\delta_1 = u_2$, $\delta_2 = u_3 - u_2^2$, $\delta_3 = u_4 - 3u_3u_2 + 2u_2^3$. Note that, except for the shift, the relation between u_n and δ_n is the same as the relation between the moments of a distribution and its cumulants, or between unconnected Green functions and connected Green functions. The inverse relation is obtained from

$$\phi(x) = \int_0^x \mathrm{d}t \exp\Big(\sum_{n=1}^{\infty} \delta_n \frac{t^n}{n!}\Big);$$

thus

$$u_{n+1} = \sum_{\alpha} \frac{n!(\delta_1)^{\alpha_1}\cdots(\delta_n)^{\alpha_n}}{\alpha_1!\cdots\alpha_n!(1!)^{\alpha_1}\cdots(n!)^{\alpha_n}}.$$

There is also a morphism between the noncommutative Pinter Hopf algebra and the noncommutative algebra of diffeomorphisms, which was defined in [8]. The relation between the Faà di Bruno Hopf algebra and the Connes-Moscovici algebra was also studied in [22].

6. Relation with renormalization

The renormalization of time-ordered products in configuration space was first considered by Bogoliubov, Shirkov and Parasiuk in [3, 5, 4] and presented in detail in the textbook [2]. It was elaborated more precisely in [20] and received its Hopf algebraic formulation in [35]. This approach was found particularly convenient for defining quantum field theories in curved spacetime [10, 11, 27, 28]. Here we show that the renormalization defined by Bogoliubov and Shirkov (see [2], section 26) or

by Pinter [35] can be obtained from our construction. We first define the bialgebra of fields $\mathcal{B}$.

6.1. **The bialgebra of fields.** We consider a finite set $\mathcal{D}$ of distinct points in $\mathbb{R}^d$. The bialgebra $\mathcal{B}$ is generated as a vector space over $\mathbb{C}$ by the symbols $\phi^n(x)$, where n is a nonnegative integer and $x \in \mathcal{D}$. The basis elements $\phi^n(x)$ are called Wick monomials in the physical literature. The algebra product of $\mathcal{B}$ is defined by $\phi^n(x) \cdot \phi^m(y) = \delta_{x,y}\phi^{n+m}(x)$ and its unit is $1_{\mathcal{B}} = \sum_{x\in\mathcal{D}} \phi^0(x)$.

The coproduct of $\mathcal{B}$ is the binomial coproduct [33]

$$\delta_{\mathcal{B}}\phi^n(x) \;=\; \sum_{k=0}^{n} \binom{n}{k} \phi^k(x) \otimes \phi^{n-k}(x),$$

and the counit is $\varepsilon_{\mathcal{B}}(\phi^n(x)) = \delta_{n,0}$. We define the degree of $\phi^n(x)$ to be n. The algebra $\mathcal{B}$ is thus a graded commutative and cocommutative bialgebra. It is infinite dimensional but finite dimensional in each degree.

The symmetric algebra $S(\mathcal{B})$ has coproduct δ induced by the coproduct of $\mathcal{B}$. More explicitly, the coproduct of $a = \{\phi^{n_1}(x_1), \dots, \phi^{n_m}(x_m)\}$ is

$$\begin{aligned} \delta a \;\; = \;\; & \sum_{i_1=0}^{n_1} \cdots \sum_{i_m=0}^{n_m} \binom{n_1}{i_1} \cdots \binom{n_m}{i_m} \\ & \{\phi^{i_1}(x_1), \dots, \phi^{i_m}(x_m)\} \otimes \{\phi^{n_1-i_1}(x_1), \dots, \phi^{n_m-i_m}(x_m)\}. \end{aligned}$$

The counit is defined by $\epsilon(1) = 1$ and $\epsilon(\phi^n(x)) = \delta_{n,0}$, and extended to $S(\mathcal{B})$ by linearity and multiplicativity. This coproduct and counit turn $S(\mathcal{B})$ into a commutative and cocommutative biagebra.

Remarks: (i) If we restrict the definition to a single point x, the commutative and cocommutative algebra $\mathcal{B}$ can be identified with the algebra of Wick monomials at $x \in \mathcal{D}$ [10, 11]. (ii) The product in $S(\mathcal{B})$ is the usual normal product of quantum field theory [37]. In other words, $\{\phi^{n_1}(x_1), \dots, \phi^{n_m}(x_m)\}$ would be written $:\!\phi^{n_1}(x_1) \cdots \phi^{n_m}(x_m)\!:$ in a quantum field theory textbook. The product in $S(\mathcal{B})$ can also be considered as a product of classical fields [19]. (iii) In quantum field theory, the counit of $S(\mathcal{B})$ is called the vacuum expectation value [9]: $\epsilon(a) = \langle 0|a|0\rangle$ for $a \in S(\mathcal{B})$. (iv) Considering a finite number of points $\mathcal{D}$ instead of all the points of $\mathbb{R}^d$ is consistent with the framework of perturbative renormalization and with the fact that the renormalization Hopf algebra encapsulates the combinatorics of renormalization but not its analytical aspects. Taking all the points of $\mathbb{R}^d$ would make $\mathcal{B}$ a non-locally-compact Hopf algebra, i.e. an object very difficult to handle.

6.2. **Time-ordered product.** To define the time-ordered product, we start from a linear map $t : S(\mathcal{B}) \to \mathbb{C}$ such that $t(1) = 1$. The time-ordered product T is a linear map $S(\mathcal{B}) \to S(\mathcal{B})$ defined by

$$T(a) \;=\; \sum t(a_{(1)})a_{(2)}, \tag{13}$$

Note that $t(a)$ can be recovered from $T(a)$ by the relation $t(a) = \epsilon\big(T(a)\big)$.

Remarks: (i) In quantum field theory, the map t is defined in terms of Feynman diagrams [6], but the combinatorics of renormalization does not depend on the precise structure of t. (ii) Equation 13 was essentially given in the paper by Epstein

and Glaser [20]. In the physics literature, it is written [10, 20]

$$T\big(\phi^{n_1}(x_1)\cdots\phi^{n_m}(x_m)\big) \;=\; \sum_{i_1=0}^{n_1}\cdots\sum_{i_m=0}^{n_m}\binom{n_1}{i_1}\cdots\binom{n_m}{i_m} \\ \langle 0|T\big(\phi^{i_1}(x_1)\cdots\phi^{i_m}(x_m)\big)|0\rangle{:}\phi^{n_1-i_1}(x_1)\cdots\phi^{n_m-i_m}(x_m){:}.$$

6.3. **Relation between time-ordered products.** According to Bogoliubov and Shirkov [2], renormalization can be seen as a particular kind of transformation from a time-ordered product T defined by a map t to a time-ordered product $\tilde{T}$ defined by a map $\tilde{t}$. If $a = \{\phi^{n_1}(x_1),\dots,\phi^{n_m}(x_m)\}$, in standard quantum field theory $t(a)$ is defined in terms of regularized Feynman propagators and $\tilde{t}(a)$ supplies the counterterms that remove the singularities of $t(a)$; in the Epstein-Glaser approach the Feynman propagators are not regularized and $t(a)$ is well-defined only when all spacetime points x_i are different, then $\tilde{t}$ is the extension of t to the case of coinciding spacetime points. It is also common in physics to consider renormalization where t and $\tilde{t}$ are both well-defined. This finite renormalization determines the effect of a change of the parameters (mass, coupling constant) describing the physical system.

The relation between time-ordered products T and $\tilde{T}$ was discussed in [35, 2, 20, 11, 27, 28, 36]. We shall follow the presentation given by Pinter [35]. She considers linear maps $O : S(\mathcal{B}) \to \mathcal{B}$. Apart from linearity, the only specific property of the maps O is the fact that they are diagonally supported. This expresses the local nature of renormalization and means that $O\big(\{\phi^{n_1}(x_1),\dots,\phi^{n_m}(x_m)\}\big)$ is zero if the relation $x_1 = x_2 = \cdots = x_m$ is not satisfied. The other properties of O will be consequences of the fact that T and $\tilde{T}$ are time-ordered products. In [35], Pinter uses the notation Δ for our O, but we changed to O (as in [28]) to avoid confusion with the coproduct. The elements $O(a)$ are called *quasilocal operators* and denoted by Λ or Δ by Bogoliubov and Shirkov [2]. The purpose of this section is a description of $O(a)$ in terms of the coproduct of $S(\mathcal{B})$ and to show how the definition of O must be modified to make it consistent with our algebraic approach.

The map O is used to define $\tilde{T}$ from T. Equation 13 of Pinter's paper [35] can be written (see also [2])

$$\tilde{T}(a) \;=\; \sum_{\pi\in\Pi_m} T\big(O(b^1)\vee\cdots\vee O(b^k)\big), \tag{14}$$

where we use the notation of Equation 9: if $a = \{\phi^{n_1}(x_1),\dots,\phi^{n_m}(x_m)\}$ is an element of $S(\mathcal{B})$, we let $b = (\phi^{n_1}(x_1),\dots,\phi^{n_m}(x_m)) \in T(\mathcal{B})$, and for π a partition with blocks $B_1,\dots B_k$, we have $b^i = \{b|B_i\}$, for $1 \le i \le k$. For notational convenience, we identify $S^1(\mathcal{B})$ with $\mathcal{B}$ in the rest of the section.

We derive now some additional properties of O. In standard quantum field theory, a single vertex is not renormalized [2]. Thus $\tilde{T}(a) = T(a) = a$ if $a \in \mathcal{B}$. This enables us to show that $O(a) = a$ if $a \in \mathcal{B}$: If $a \in \mathcal{B}$, then the sum in Equation 14 has only one term, corresponding to the partition $\{\{1\}\}$ of the set $\{1\}$, and thus $\tilde{T}(a) = T\big(O(a)\big)$. But $O(a) \in \mathcal{B}$ (by definition of O), and thus $T\big(O(a)\big) = O(a)$. The fact that $\tilde{T}(a) = a$ implies that $O(a) = a$.

If $a = \{\phi^{n_1}(x_1),\dots,\phi^{n_m}(x_m)\}$ with $m > 1$, we use $O(\{\phi^{n_i}(x_i)\}) = \{\phi^{n_i}(x_i)\}$ to rewrite Equation 14

$$\tilde{T}(a) \;=\; T(a) + T(O(a)) + \sum_{\pi\in\Pi'_m} T\big(O(b^1)\vee\cdots\vee O(b^k)\big),$$

where Π'_m indicates set of all partitions of $\{1,\dots,m\}$, except for $\pi = \{\{1\},\dots,\{m\}\}$ and $\pi = \{\{1,\dots,m\}\}$. But $O(a) \in \mathcal{B}$ and T acts as the identity on $\mathcal{B}$, thus

$$\tilde{T}(a) \;=\; T(a) + O(a) + \sum_{\pi\in\Pi'_m} T\big(O(b^1)\vee\cdots\vee O(b^k)\big). \tag{15}$$

From this transformation formula and the support property of O we deduce

Proposition 4. *If $a = \{\phi^{n_1}(x_1),\dots,\phi^{n_m}(x_m)\}$, then $O(a) = \sum c(a_{(1)})a_{(2)}$, where $c(a) = \epsilon\big(O(a)\big)$. Moreover, if $m = 1$, then $c(a) = \epsilon(a)$, if $m > 1$, then $c(a)$ is supported on $x_1 = \cdots = x_m$ and can be obtained recursively from $\tilde{t}$ and t by*

$$c(a) = \tilde{t}(a) - t(a) - \sum_{\pi\in\Pi'_m} \sum c(b^1_{(1)})\cdots c(b^k_{(1)})t(b^1_{(2)}\vee\cdots\vee b^k_{(2)}). \tag{16}$$

Proof. The proof is by induction. Take $a = \{\phi^{n_1}(x_1),\dots,\phi^{n_m}(x_m)\}$. If $m = 1$, we have $O(a) = a$, so that $O(a) = \sum \epsilon(a_{(1)})a_{(2)}$ and $c(a) = \epsilon(a)$. For $m = 2$, the set Π'_m is empty, and thus Equation 15 yields $\tilde{T}(a) = T(a) + O(a)$. We know that $T(a) = \sum t(a_{(1)})a_{(2)}$ and $\tilde{T}(a) = \sum \tilde{t}(a_{(1)})a_{(2)}$, thus $O(a) = \sum c(a_{(1)})a_{(2)}$ with $c = \tilde{t} - t$. Now assume that the proposition is true up to $m-1$ and take $a = \{\phi^{n_1}(x_1),\dots,\phi^{n_m}(x_m)\}$. In Equation 15, all $O(b^i)$ can be written $\sum c(b^i_{(1)})b_{(2)}$ because the degree of b^i is smaller than m. Hence, we can write

$$\tilde{T}(a) \;=\; T(a) \;+\; O(a) \;+\; \sum_{\pi\in\Pi'_m} \sum c(b^1_{(1)})\cdots c(b^k_{(1)})T\big(b^1_{(2)}\vee\cdots\vee b^k_{(2)}\big).$$

Equation 13 now yields

$$\begin{aligned}\sum \tilde{t}(a_{(1)})a_{(2)} \;&=\; \sum t(a_{(1)})a_{(2)} \;+\; O(a)\\ &\quad+\sum_{\pi\in\Pi'_m} \sum c(b^1_{(1)})\cdots c(b^k_{(1)})t\big(b^1_{(2)}\vee\cdots\vee b^k_{(2)}\big)b^1_{(3)}\vee\cdots\vee b^k_{(3)}.\end{aligned}$$

Since $a = b^1\vee\cdots\vee b^k$, the factor $b^1_{(3)}\vee\cdots\vee b^k_{(3)}$ can be written as $a_{(3)}$, and Equation 16 follows from the coassociativity of the coproduct.

The fact that $O(a)$ is supported on $x_1 = \cdots = x_m$ implies that $c(a) = \epsilon\big(O(a)\big)$ is supported on $x_1 = \cdots = x_m$. Thus, $c(a) = f(x_1)\delta_{x_2,x_1}\cdots\delta_{x_m,x_1}$, where f is some function of x_1. In flat spacetime and in the absence of an external field, the system is translation invariant and $f(x_1)$ is a constant. □

Now comes a crucial step which is not apparent in the usual renormalization. In quantum field theory, when $x_1 = \cdots = x_m$, $\{\phi^{n_1}(x_1),\dots,\phi^{n_m}(x_m)\}$ is identified with

$$\prod_{p=1}^{m} \phi^{n_p}(x_1) \quad=\quad \phi^{n_1+\cdots+n_m}(x_1),$$

where the product $\prod$ is in $\mathcal{B}$. After of this identification, the expression $O(a) = \sum c(a_{(1)})a_{(2)}$ is replaced by

$$\Lambda(a) = \sum c(a_{(1)}) \prod a_{(2)}, \tag{17}$$

where the product means that, if $a = y_1\vee y_2\vee\cdots\vee y_p$ with $y_i \in \mathcal{B}$, then $\prod a = y_1\cdot y_2\cdots y_p$ where the product $\cdot$ is in $\mathcal{B}$. From our algebraic point of view, $\Lambda(a)$ is different from $O(a)$. The map Λ is more satisfactory because $\prod a_{(2)}$ belongs to $\mathcal{B}$

and it is clear that Λ maps $S(\mathcal{B})$ to $\mathcal{B}$. This was not the case with the expression $O(a) = \sum c(a_{(1)})a_{(2)}$ because $a_{(2)}$ belongs to $S(\mathcal{B})$.

Therefore, in the following, we shall use Λ instead of O to define the renormalized time-ordered product $\overline{T}$ as the linear operator $S(\mathcal{B}) \to S(\mathcal{B})$

$$\overline{T}(a) = \sum_{\pi} T\big(\Lambda(b^1) \vee \cdots \vee \Lambda(b^k)\big). \tag{18}$$

If we expand the terms $\Lambda(b^i)$ with Equation 17 we find

$$\overline{T}(a) = \sum_{\pi} \sum c(b^1_{(1)}) \cdots c(b^k_{(1)}) T\Big(\prod b^1_{(2)} \vee \cdots \vee \prod b^k_{(2)}\Big). \tag{19}$$

Although $\overline{T}(a)$ would be undistinguishable from $\tilde{T}(a)$ in quantum field theory, these two quantities are different in our algebraic approach. The main difference is the fact that there is no map $\bar{t}$ such that $\overline{T}(a) = \sum \bar{t}(a_{(1)})a_{(2)}$.

Equation 19 can be translated into the usual renormalization prescription by saying that each b^i is a generalized vertex in the sense of Bogoliubov and Shirkov [2], the operation $\prod b^i_{(2)}$ shrinks the points of $b^i_{(2)}$ into a single point leaving the external lines unchanged and the number $c(b^i_{(1)})$ describes the counterterm associated to the generalized vertex.

6.4. **Relation to the renormalization coproduct.** It remains to relate the last result to the renormalization coproduct. If we compare expression 19 to the commutative renormalization coproduct (9), we see that

$$\overline{T}(a) = \sum C(a_{[1]}) T(a_{[2]}), \tag{20}$$

where $C(a) = c(a)$ for $a \in S(\mathcal{B})^+$ and $C(uv) = C(u)C(v)$ for $u, v \in S(S(\mathcal{B})^+)$, and where we have used the alternate Sweedler notation $\Delta(a) = \sum a_{[1]} \otimes a_{[2]}$ for the commutative renormalization coproduct. The quantum field relation $\tilde{T}(a) = \sum \tilde{t}(a_{(1)})a_{(2)}$ becomes

$$\overline{T}(a) = \sum C(a_{[1]}) t(a_{[2](1)}) a_{[2](2)}.$$

Renormalization can be seen from (at least) two points of view: (i) as a way to transform a time-ordered product T into a renormalized time-ordered product $\overline{T}$, (ii) as a way to transform a bare Lagrangian (i.e. an element a of $\mathcal{B}$) into a renormalized Lagrangian. We establish now the connection between these two points of view. If $a \in \mathcal{B}$ we define $a^n \in S^n(\mathcal{B})$ by $a^n = a \vee \cdots \vee a$ (n times). With this notation we can define, in the sense of formal power series in a complex variable λ, the series $\mathrm{e}^{\lambda a} = \sum \lambda^n a^n / n!$. We have

Proposition 5. *If a is an element of $\mathcal{B}$ and λ a complex variable, then*

$$\overline{T}\big(\mathrm{e}^{\lambda a}\big) = T\big(\mathrm{e}^{\lambda \bar{a}(\lambda)}\big), \tag{21}$$

where

$$\bar{a}(\lambda) = \sum_{n=1}^{\infty} \frac{\lambda^{n-1}}{n!} \Lambda(a^n) = \Lambda\Big(\frac{\mathrm{e}^{\lambda a} - 1}{\lambda}\Big). \tag{22}$$

In this proposition, Equations 21 and 22 are understood in the sense of formal power series in λ. The first term of $\bar{a}(\lambda)$ is $\Lambda(a) = a$, which is called the bare Lagrangian in quantum field theory. The next terms are called the counterterms of the Lagrangian. The proof of the proposition is straighforward:

Proof. We expand $\overline{T}\big(\mathrm{e}^{\lambda a}\big)$ as

$$\overline{T}\big(\mathrm{e}^{\lambda a}\big) = 1 + \sum_{n=1}^{\infty} \frac{\lambda^n}{n!} \overline{T}(a^n).$$

To calculate $\overline{T}(a^n)$ we use Equation 19, noting that in this case the elements $b^1, \dots, b^k \in S(\mathcal{B})$ depend only on the sizes of the blocks of π. The number of partitions of n different objects with α_i blocks of size i was given in Section 4.2. This gives us

$$\overline{T}(a^n) = \sum_{\alpha} \frac{n! T\big(\Lambda(a^1)^{\alpha_1} \vee \cdots \vee \Lambda(a^n)^{\alpha_n}\big)}{\alpha_1! \alpha_2! \cdots \alpha_n! (1!)^{\alpha_1} (2!)^{\alpha_2} \cdots (n!)^{\alpha_n}},$$

where the n-tuples α are described in section 4.1. Consider now $g(\lambda) = \lambda \bar{a}(\lambda)$. This is a formal exponential power series with coefficients $g_n = \Lambda(a^n)$. Thus, $T\big(\mathrm{e}^{\lambda \bar{a}(\lambda)}\big) = 1 + T\big(\mathrm{e}^{g(\lambda)} - 1\big) = 1 + T\big(f(g(\lambda))\big)$, where $f(\lambda) = \mathrm{e}^{\lambda} - 1$ is an exponential power series with coefficients $f_n = 1$. If we use the Faà di Bruno formula for the composition of series, we obtain the term of degree λ^n in the exponential series $f(g(\lambda))$ as

$$\sum_{\alpha} \frac{n! \Lambda(a^1)^{\alpha_1} \vee \cdots \vee \Lambda(a^n)^{\alpha_n}}{\alpha_1! \alpha_2! \cdots \alpha_n! (1!)^{\alpha_1} (2!)^{\alpha_2} \cdots (n!)^{\alpha_n}}.$$

Therefore, the linearity of T implies that

$$\overline{T}\big(\mathrm{e}^{\lambda a}\big) = 1 + \sum_{n=1}^{\infty} \frac{\lambda^n}{n!} \overline{T}(a^n) = 1 + T\big(f(g(\lambda))\big) = T\big(\mathrm{e}^{\lambda \bar{a}(\lambda)}\big).$$

□

We have thus shown that the renormalization coproduct gives the same result as the Bogoliubov-Shirkov-Epstein-Glaser renormalization. On the other hand, it was proved in [20] and [34] that the latter renormalization coincides with the standard BPHZ renormalization. Thus, the renormalization coproduct recovers the standard renormalization of scalar fields.

The coassociativity of the renormalization coproduct means that the transitions from one renormalized time-ordered product to another can be composed associatively. In other words, if $\overline{T}(a) = \sum C(a_{[1]}) T(a_{[2]})$ and $\overline{T}'(a) = \sum C'(a_{[1]}) \overline{T}(a_{[2]})$, then $\overline{T}'(a) = \sum C''(a_{[1]}) T(a_{[2]})$ with $C''(a) = \sum C'(a_{[1]}) C(a_{[2]})$.

In this section we have assumed that $T\big(\phi^n(x)\big) = \phi^n(x)$. This is consistent with the use of the Pinter algebra. It was pointed out by Hollands and Wald [27, 28] that renormalization at a point is necessary in curved spacetime and is given by

$$\Lambda(\phi^n(x)) \;=\; \sum_{k=0}^{n} \binom{n}{k} c\big(\phi^k(x)\big) \phi^{n-k}(x).$$

This is exactly Equation 17. Thus, the renormalization of quantum field theory in curved spacetimes requires the commutative renormalization bialgebra instead of the Pinter Hopf algebra.

Many equations of this section are valid for any commutative bialgebra $\mathcal{B}$: the definition (13) of the time-ordered product is the right coregular action of $S(\mathcal{B})^*$ on $S(\mathcal{B})$; for $c \in S(\mathcal{B})^*$, Equation 17 defines a map $\Lambda : S(\mathcal{B}) \to \mathcal{B}$, and the renormalized time-ordered product (18) is then well defined and satisfies Proposition 5.

Remark. If we take the realistic example of a quantum field theory with the interaction $a = \int \phi(x)^4 \mathrm{d}x$ in four spacetime dimensions (in our framework, the integral should be replaced by a finite sum), the quasilocal operators take the explicit form [2]:

$$\Lambda(a^n) \quad = \quad C_1^n \int \phi^2(x)\mathrm{d}x + C_2^n \int \phi^4(x)\mathrm{d}x + C_3^n \int \phi(x)\Box\phi(x)\mathrm{d}x,$$

where C_i^n are constants. In this equation, the first two quasilocal operators renormalize logarithmic divergences and are of the type treated in this paper. The third quasilocal operator involves derivatives of the fields. It is used to remove quadratic divergences and is absent in our approach. In other words, the present renormalization algebra can only deal with logarithmic divergences. To take care of higher divergences we should need to study the interplay between the renormalization algebra and derivations of this algebra. However, even at the quantum field level, the interplay between time-ordered products and derivatives is a delicate matter [17, 18, 19]. Its Hopf algebraic interpretation is still an open problem. Within the Connes-Kreimer approach, this problem was solved in terms of an "external structure" [12].

7. Conclusion

We have constructed the renormalization bialgebra corresponding to any bialgebra $\mathcal{B}$. In standard quantum field theory, the commutative Pinter Hopf algebra is generally used, but the noncommutative one may be relevant to the renormalization of some noncommutative quantum field theories [38].

At the mathematical level, the renormalization Hopf algebra found unexpected applications in number theory [21]. Moreover, an intriguing connection was observed between the renormalization bialgebra $T(T(\mathcal{B})^+)$ and a construction involving operads [42, 41]. Such a connection is another manifestation of the deep mathematical meaning of renormalization. This operad is a realization of Kreimer's suggestion that operads should play a role in renormalization theory [31]. Kreimer himself defined an operad of renormalization based on Feynman diagrams [32].

8. Acknowledgements

We are very grateful to Gudrun Pinter for the detailed information on her results that she kindly gave us. We wish to thank Alessandra Frabetti, Muriel Livernet, Klaus Fredenhagen, Romeo Brunetti, Robert Oeckl, José M. Gracia-Bondia, Michael Dütsch, Ieke Moerdijk, Pepijn van der Laan, Bertfried Fauser, Frédéric Patras and Christian Voigt for helpful comments and discussions. A special thanks to Jean-Louis Loday for his constant help and encouragement. Finally, we thank the referee for many valuable comments and suggestions.

References

1. M. Abramowitz and I.A. Stegun, *Handbook of mathematical functions, with formulas, graphs, and mathematical tables*, fifth ed., Dover, New York, 1964.
2. N.N. Bogoliubov and D.V. Shirkov, *Introduction to the theory of quantized fields*, Interscience Pub. Inc., New York, 1959.
3. N.N. Bogoliubov and D.W. Shirkov, *(in Russian)*, Uzpekhi fiz. Nauk **57** (1955), 3–91.
4. N.N. Bogoljubow and O.S. Parasiuk, *Über die Multiplikation der Kausalfunktionen in der Quantentheorie der Felder*, Acta Math. **97** (1957), 227–266.

5. N.N. Bogoljubow and D.W. Schirkow, *Probleme der Quantenfeldtheorie. II. Beseitigung der Divergenzen aus der Streumatrix*, Fort. d. Phys. **4** (1956), 438–517.
6. Ch. Brouder, *Quantum groups and interacting quantum fields*, Group 24: Physical and Mathematical Aspects of Symmetries (Bristol) (J.-P. Gazeau, R. Kerner, J.-P. Antoine, S. Métens, and J.-Y. Thibon, eds.), Institute of Physics Publishing, 2003, pp. 787–790.
7. Ch. Brouder, B. Fauser, A. Frabetti, and R. Oeckl, *Quantum field theory and Hopf algebra cohomology*, J. Phys. A: Math. Gen. **34** (2004), 5895–5927.
8. Ch. Brouder, A. Frabetti, and Ch. Krattenthaler, *Noncommutative Hopf algebra of formal diffeomorphisms*, Adv. Math. **200** (2006), 479-524.
9. Ch. Brouder and R. Oeckl, *Quantum groups and quantum field theory: The free scalar field*, Mathematical Physics Research on the Leading Edge, (Hauppauge NY) (C.V. Benton, ed.), Nova Science, 2004, pp. 63–90.
10. R. Brunetti and K. Fredenhagen, *Interacting quantum fields in curved space: Renormalization of ϕ^4*, Operator Algebras and Quantum Field Theory (Cambridge) (S. Doplicher, R. Longo, J.E. Roberts, and L. Zsido, eds.), International Press of Boston, 1997, pp. 546–563.
11. ———, *Microlocal analysis and interacting quantum field theories: renormalization on physical backgrounds*, Commun. Math. Phys. **208** (2000), 623–661.
12. A. Connes and D. Kreimer, *Renormalization in quantum field theory and the Riemann-Hilbert problem I: the Hopf algebra structure of graphs and the main theorem*, Commun. Math. Phys. **210** (2000), 249–273.
13. A. Connes and H. Moscovici, *Hopf algebras, cyclic cohomology and the transverse index theorem*, Commun. Math. Phys. **198** (1998), 199–246.
14. S. Dăscălescu, C. Năstăsescu, and Ş. Raianu, *Hopf algebras*, Marcel Dekker Inc., New York, 2001.
15. F. Faà di Bruno, *Sullo svilupo delle funzioni*, Ann. Sci. Mat. Fis. **6** (1855), 479–480.
16. P. Doubilet, *A Hopf algebra arising from the lattice of partitions of a set*, J. Alg. **28** (1974), 127–132.
17. M. Dütsch and F.-M. Boas, *The master Ward identity*, Rev. Math. Phys. **14** (2002), 977–1049.
18. M. Dütsch and K. Fredenhagen, *The master Ward identity and generalized Schwinger-Dyson equation in classical field theory*, Commun. Math. Phys. **243** (2003), 275–314.
19. ———, *Causal perturbation theory in terms of retarded products, and a proof of the Action Ward Identity*, Rev. Math. Phys. **16** (2004), 1291–1348.
20. H. Epstein and V. Glaser, *The role of locality in perturbation theory*, Ann. Inst. Henri Poincaré **19** (1973), 211–295.
21. B. Fauser, *The Dirichlet Hopf algebra of arithmetics*, J. Knot Theory Ramifications **16** (2007), 1–60.
22. H. Figueroa and J.M. Gracia-Bondia, *Combinatorial Hopf algebras in quantum field theory I*, Rev. Math. Phys. **17** (2005), 881–976.
23. J.M. Gracia-Bondia and S. Lazzarini, *Connes-Kreimer-Epstein-Glaser renormalization*, (2000), arXiv:hep-th/0006106.
24. J.M. Gracia-Bondia, J.C. Varilly, and H. Figueroa, *Elements of noncommutative geometry*, Birkhäuser, Boston, 2001.
25. M. Haiman and W. Schmitt, *Incidence algebra antipodes and Lagrange inversion in one and several variables*, J. Combin. Theor. A **50** (1989), 172–185.
26. F. Hivert, *Combinatoire des fonctions quasi-symétriques*, Ph.D. thesis, Marne-la-Vallée University, 1999.
27. S. Hollands and R.M. Wald, *Local Wick polynomials and time ordered products of quantum fields in curved spacetime*, Commun. Math. Phys. **223** (2001), 289–326.
28. ———, *On the renormalisation group in curved spacetime*, Commun. Math. Phys. **237** (2003), 123–160.
29. S.A. Joni and G.-C. Rota, *Coalgebras and bialgebras in combinatorics*, Stud. Appl. Math. **61** (1979), 93–139.
30. Ch. Kassel, *Quantum groups*, Springer Verlag, New York, 1995.
31. D. Kreimer, 1999, Personal communication.
32. ———, *Combinatorics of (perturbative) quantum field theory*, Phys. Repts. **363** (2002), 387–424.
33. S. Majid, *Foundations of quantum group theory*, Cambridge University Press, Cambridge, 1995.

34. G. Pinter, *Epstein-Glaser renormalization: Finite renormalizations, the S-matrix of ϕ^4 theory and the action principle*, Ph.D. thesis, Hamburg University, 2000.
35. ———, *The Hopf algebra structure of Connes and Kreimer in Epstein-Glaser renormalization*, Lett. Math. Phys. **54** (2000), 227–233.
36. ———, *Finite renormalization in the Epstein-Glaser framework and renormalization of the S-matrix of Φ^4-theory*, Ann. Phys. (Leipzig) **10** (2001), 333–363.
37. M. Reed and B. Simon, *Methods of modern mathematical physics II: Fourier analysis, self-adjointness*, Academic Press, New York, 1975.
38. W.G. Ritter, *Description of noncommutative theories and matrix models by Wightman functions*, J. Math. Phys. **45** (2004), 4980–5002.
39. W. Schmitt, *Antipodes and incidence coalgebras*, J. Combin. Theor. A **46** (1987), 264–290.
40. ———, *Incidence Hopf algebras*, J. Pure Appl. Alg. **96** (1994), 299–330.
41. P. van der Laan, *Operads and the Hopf algebras of renormalisation*, (2003), math-ph/0311013.
42. P. van der Laan and I. Moerdijk, *The renormalisation bialgebra and operads*, (2002), hep-th/0210226.

Institut de Minéralogie et de Physique des Milieux Condensés, CNRS UMR7590, Universités Paris 6 et 7, IPGP, 140 rue de Lourmel, 75015 Paris, France. E-mail: christian.brouder@impmc.jussieu.fr

Department of Mathematics, The George Washington University, 1922 F St. NW, Washington, DC 20052, USA. E-mail: wschmitt@gwu.edu