Quantum groups and quantum field theory: I. The free scalar field

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Abstract. The quantum field algebra of real scalar fields is shown to be an example of infinite dimensional quantum group. The underlying Hopf algebra is the symmetric algebra $S(V)$ and the product is Wick's normal product. Two coquasitriangular structures can be built from the two-point function and the Feynman propagator of scalar fields to reproduce the operator product and the time-ordered product as twist deformations of the normal product. A correspondence is established between the quantum group and the quantum field concepts. On the mathematical side the underlying structures come out of Hopf algebra cohomology.

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1. Introduction

This paper brings together quantum groups and quantum field theory. More precisely, it shows that quantum field operator algebras are infinite dimensional quantum groups. A quantum group is a relatively recent mathematical structure that has attracted considerable interest in mathematics and physics. It is a Hopf algebra equipped with a coquasitriangular structure||. Besides the quantum group structure for the field algebra, many other concepts of quantum field theory can be translated into the quantum group language. For instance, the normal product is the Hopf algebra product, the expectation value over the vacuum is the counit, the operator product and time-ordered product are twisted products.

This paper presents the basic concepts of quantum groups and their applications to the free real scalar field. Fermions and interacting fields will be presented in forthcoming publications. We try to familiarize the reader with the notions of Hopf algebra, Laplace pairing, coquasitriangular structure and twisted product. Results are derived in full detail.

In quantum field theory, the difference between a product of two field operators and their normal product is a scalar. Therefore, any product of field operators can be written as a linear combination of normal products of operators. We denote by $S(V)$ the vector space spanned by all normal products of operators. Similarly, the difference between the time-ordered product of two operators and their normal product is a scalar function. Thus, all time-ordered products of operators belong also to $S(V)$. In other words, quantum field theory can be entirely discussed in the space $S(V)$ of normal products. Working in $S(V)$ has several advantages, already noted by Houriet and Kind [2] and later stressed by Wick [3] and by Gupta [4]: $S(V)$ is equipped with a commutative product (the normal product) and the expectation value of an element u of $S(V)$ over the vacuum is zero if u has no scalar component.

This motivates us to use the Hopf algebra structure of $S(V)$. The product of $S(V)$ is the normal product while the counit describes the vacuum expectation value. Remarkably, both the operator product and the time-ordered product arise now as twist deformations of the normal ordered product. More precisely, we first make $S(V)$ into a quantum group by adding a Laplace pairing (or coquasitriangular structure). The Laplace pairing is determined by appropriate Green functions or propagators. This Laplace pairing then gives rise to a new product on $S(V)$, called *twisted product* (a special case of Sweedler's crossed product [5]), which turns out to reproduce the operator or the time-ordered product, depending on the chosen Green function. In this sense, quantum groups enable us to define second quantization without commutators. For the time-ordered product we also obtain the time ordering prescription T linking the normal and time-ordered product as an algebra automorphism $T : S(V) \to S(V)$.

Note that our approach here is closely related to the one of braided quantum field theory [6] which employs a path integral picture. The starting point there is the free tensor algebra over the classical fields to be compared with $S(V)$ in our context. (There, the analog of our coquasitriangular structure is the map γ.) Furthermore, the connection between a twisted product and Wick's theorem was already proposed by Fauser in a fermionic (i.e. antisymmetric) setting [7]. In general, the quantum group framework enables us to replace combinatorial proofs of quantum field theory by algebraic manipulations. Several examples of this were given in reference [8].

|| More frequently, the quantum groups are defined with a quasitriangular structure R but the two notions are dual to one another, see [1].

This paper is broadly divided into two parts. In the first part (Sections 2 and 3) we introduce the required mathematical structures while the second applies them to quantum field theory (Section 4). We start in Section 2 with the definition of the Hopf algebra $S(V)$ and a Laplace pairing. In Section 3 we proceed to introduce the twisted product on $S(V)$, based on the Laplace pairing. Furthermore, in the case where the twisted product is commutative, we construct the algebra automorphism $T : S(V) \to S(V)$ which relates the twisted and untwisted products. We also show that this T can be expressed as an exponential. In Section 4 we turn to quantum field theory. First, we conveniently define the vector space V of smoothed field operators on a Fock space. $S(V)$ is then identified as the Hopf algebra of polynomials in the operators with normal product. Using appropriate Green functions we define Laplace pairings which give rise to the operator and time-ordered products. The proof that we indeed obtain these products as twisted products is performed by invoking Wick's theorem. For the time-ordered product the T-map gives a compact expression for the vacuum expectation value of the free field theory. Finally, the conclusion describes some prospects of the present work. The appendix provides a lightning introduction to Hopf algebras.

2. The symmetric quantum group

A quantum group is a Hopf algebra with a (co)quasitriangular structure and we describe now the Hopf algebra that we need in the scalar quantum field theory: the symmetric Hopf algebra $S(V)$. Readers who are not familiar with Hopf algebras should first read the appendix.

2.1. The symmetric Hopf algebra

Later, the symmetric product will be used to describe the normal product of bosonic creation and annihilation operators (see Section 4.2) but we define now the symmetric Hopf algebra for a general vector space V over $\mathbb{C}$. The symmetric algebra $S(V)$ is the direct sum

$$S(V) = \bigoplus_{n=0}^{\infty} S^n(V),$$

where $S^0(V) = \mathbb{C}$ and $S^1(V) = V$. For any vector space V, $S(V)$ is the free unital commutative and associative algebra over V [9]. If V is finite dimensional with basis e_i, $S^n(V)$ is spanned by elements of the form $e_{i_1} \vee \cdots \vee e_{i_n}$, with $i_1 \leq i_2 \leq \ldots \leq i_n$. The symbol $\vee$ denotes the associative and commutative product $\vee : S^m(V) \otimes S^n(V) \longrightarrow S^{m+n}(V)$. For finite dimensional V, the product is defined on monomials by $(e_{i_1} \vee \cdots \vee e_{i_m}) \vee (e_{i_{m+1}} \vee \cdots \vee e_{i_{m+n}}) = e_{i_{\sigma(1)}} \vee \cdots \vee e_{i_{\sigma(m+n)}}$, where σ is the permutation on $m+n$ elements such that $i_{\sigma(1)} \leq \cdots \leq i_{\sigma(m+n)}$ and then extended by linearity and associativity to all elements of $S(V)$. The unit of $S(V)$ is the scalar unit $1 \in S^0(V)$ (i.e. for any $u \in S(V)$: $1 \vee u = u \vee 1 = u$). The algebra $S(V)$ is graded, the degree of an element u of $S^n(V)$ is $|u| = n$. In the finite dimensional case, $S(V)$ can be seen as the algebra of polynomials in the variables $\{e_i\}$, the elements of $S^n(V)$ being homogeneous polynomials of degree n.

In this paper, u, v, w designate elements of $S(V)$, a, a_i, b, c elements of V.

The coproduct of $S(V)$ is defined as follows

$$\Delta 1 \qquad = 1 \otimes 1,$$

$$\Delta a \quad = a\otimes 1 + 1\otimes a \quad \text{for} \quad a\in V, \tag{1}$$
$$\Delta(u\vee v) = \sum (u_{(1)}\vee v_{(1)})\otimes(u_{(2)}\vee v_{(2)}), \tag{2}$$

where Sweedler's notation was used for the coproduct of u and v: $\Delta u = \sum u_{(1)}\otimes u_{(2)}$ and $\Delta v = \sum v_{(1)}\otimes v_{(2)}$. This coproduct is coassociative and cocommutative. As an example, we calculate the coproduct $\Delta(a\vee b)$, with a and b in V. From equation (2)

$$\Delta(a\vee b) = \sum (a_{(1)}\vee b_{(1)})\otimes(a_{(2)}\vee b_{(2)}),$$

From the definition (1) of the coproduct acting on elements of V we know that $\Delta a = a\otimes 1 + 1\otimes a$ and $\Delta b = b\otimes 1 + 1\otimes b$. Thus,

$$\begin{aligned}\Delta(a\vee b) &= \sum (a_{(1)}\vee b_{(1)})\otimes(a_{(2)}\vee b_{(2)})\\ &= (a\vee b)\otimes(1\vee 1) + (a\vee 1)\otimes(1\vee b) + (1\vee b)\otimes(a\vee 1)\\ &\quad + (1\vee 1)\otimes(a\vee b)\\ &= (a\vee b)\otimes 1 + a\otimes b + b\otimes a + 1\otimes(a\vee b).\end{aligned}$$

At the next degree, we have

$$\begin{aligned}\Delta(a\vee b\vee c) &= 1\otimes a\vee b\vee c + a\otimes b\vee c + b\otimes a\vee c + c\otimes a\vee b\\ &\quad + a\vee b\otimes c + a\vee c\otimes b + b\vee c\otimes a + a\vee b\vee c\otimes 1.\end{aligned}$$

Generally, if $u = a_1\vee a_2\vee\ldots\vee a_n$, the coproduct of u can be written explicitly as ([10] p.450)

$$\begin{aligned}\Delta u &= u\otimes 1 + 1\otimes u\\ &\quad + \sum_{p=1}^{n-1}\sum_{\sigma} a_{\sigma(1)}\vee\ldots\vee a_{\sigma(p)}\otimes a_{\sigma(p+1)}\vee\ldots\vee a_{\sigma(n)},\end{aligned} \tag{3}$$

where σ runs over the $(p, n-p)$-shuffles. A $(p, n-p)$-shuffle is a permutation σ of $(1,\ldots,n)$ such that $\sigma(1) < \sigma(2) < \ldots < \sigma(p)$ and $\sigma(p+1) < \ldots < \sigma(n)$.

The counit is defined by $\varepsilon(1) = 1$ and $\varepsilon(u) = 0$ if $u\in S^n(V)$ and $n > 0$. The antipode is defined by $\mathrm{s}(u) = (-1)^n u$ if $u\in S^n(V)$. In particular, $\mathrm{s}(1) = 1$. Since the symmetric product is commutative, the antipode is an algebra homomorphism: $\mathrm{s}(u\vee v) = \mathrm{s}(u)\vee \mathrm{s}(v)$.

The fact that this coproduct is coassociative and that Δ, ε and S give $S(V)$ the structure of a cocommutative Hopf algebra, are classical results (see [9, 10, 11] and references therein).

2.2. The Laplace pairing

We define a *Laplace pairing* on $S(V)$ as a bilinear form $V\times V\to\mathbb{C}$, that we denote by $(a|b)$, which is extended to $S(V)$ by $(1|1) = 1$ and the following recursions

$$(u\vee v|w) = \sum (u|w_{(1)})(v|w_{(2)}), \tag{4}$$
$$(u|v\vee w) = \sum (u_{(1)}|v)(u_{(2)}|w). \tag{5}$$

It is important to stress that we do not assume any special symmetry for the bilinear form (i.e. it is a priori neither symmetric nor antisymmetric).

These recursions have the following unique solution [12]: If $u = a_1\vee a_2\vee\ldots\vee a_k$ and $v = b_1\vee b_2\vee\ldots\vee b_n$, then $(u|v) = 0$ if $n\neq k$ and $(u|v) = \mathrm{per}(a_i|b_j)$ if $n = k$. We recall that the *permanent* of the matrix $(a_i|b_j)$ is

$$\mathrm{per}(a_i|b_j) = \sum_{\sigma} (a_1|b_{\sigma(1)})\cdots(a_k|b_{\sigma(k)}), \tag{6}$$

where the sum is over all permutations σ of $\{1,\ldots,n\}$. The permanent is a kind of determinant where all signs are positive. For instance

$$(a\vee b|c\vee d) = (a|c)(b|d) + (a|d)(b|c).$$

An important concept in the theory of quantum groups is that of a *coquasitriangular structure* [1]. This is a bilinear form on a Hopf algebra satisfying certain conditions. Indeed one may define a quantum group precisely as a Hopf algebra with a coquasitriangular structure. The conditions coincide in the case of $S(V)$ with that of a Laplace pairing, i.e. the two concepts are identical for $S(V)$. The terminology of these concepts is not settled yet in the Hopf algebra literature. A coquasitriangular structure is also called an R-form [13], a braiding [14] or a universal r-form [15], a Laplace pairing¶ is also called a duality pairing [1], a skew pairing [14], a dual pairing [16], a bicharacter [17] or a Hopf pairing [18]. We use the term "Laplace pairing" because it was introduced very early by Rota and coll. in [19] and it reflects the fact that equations (4) and (5) are an elegant way of writing the Laplace identities for permanents [20]. In the case of determinants, the Laplace identities express the determinant in terms of a sum of products of minors (see reference [21] p.93).

As an exercise, we derive the following identity:

$$(u|1) = (1|u) = \varepsilon(u). \tag{7}$$

Because of the definition of the counit and the antipode we have

$$u = \sum \varepsilon(u_{(1)})u_{(2)} = \sum \mathrm{s}(u_{(1)(1)})\vee u_{(1)(2)}\vee u_{(2)}.$$

From the fact that $\Delta 1 = 1\otimes 1$, we deduce from equation (4)

$$\begin{aligned}(u|1) &= \sum (\mathrm{s}(u_{(1)(1)})|1)(u_{(1)(2)}|1)(u_{(2)}|1)\\ &= \sum (\mathrm{s}(u_{(1)})|1)(u_{(2)(1)}|1)(u_{(2)(2)}|1) = \sum (\mathrm{s}(u_{(1)})|1)(u_{(2)}|1\vee 1)\\ &= \sum (\mathrm{s}(u_{(1)})u_{(2)}|1\vee 1) = \varepsilon(u)(1|1) = \varepsilon(u).\end{aligned}$$

3. The twisted product

This section introduces the *twisted product*. Its importance stems form the fact that it will allow us to treat in one single stroke the operator product and the time-ordered product, depending on the definition of the bilinear form $(a|b)$. We define it as

$$u\circ v = \sum u_{(1)}\vee v_{(1)}\,(u_{(2)}|v_{(2)}). \tag{8}$$

This product (for general commutative Hopf algebras) was introduced by Sweedler as a *crossed product* in Hopf algebra cohomology theory [5]. Indeed, it turns out that a Laplace pairing in our case is not only a coquasitriangular structure but also a 2-cocycle. Drinfeld showed that 2-cocycles give rise to *twist deformations* of Hopf algebras and their representation categories [22]. The twisted product can be understood as precisely such a twist deformation+. As we aim to be introductory here we shall not dwell on these facts although they are the conceptual origin of much of this section including the T-map. More details will be given in [24].

¶ For a general Hopf algebra, equation (5) becomes $(u|v\vee w) = \sum (u_{(2)}|v)(u_{(1)}|w)$ in the definition of a Laplace pairing. This is equivalent to (5) in our case of a cocommutative Hopf algebra.

+ To be precise, the twisted product is a twist deformation as a comodule algebra, not as a Hopf algebra, see [23].

By cocommutativity of the coproduct on $S(V)$, the definition (8) is equivalent to

$$\begin{aligned} u \circ v &= \sum u_{(1)} \vee v_{(2)} \, (u_{(2)}|v_{(1)}) = \sum u_{(2)} \vee v_{(1)} \, (u_{(1)}|v_{(2)}) \\ &= \sum (u_{(1)}|v_{(1)}) \, u_{(2)} \vee v_{(2)}. \end{aligned}$$

We start with the simplest example. If a and b are in V,

$$\begin{aligned} a \circ b &= (1|1) a \vee b + (a|1) 1 \vee b + (1|b) a \vee 1 + (a|b) 1 \vee 1 \\ &= a \vee b + (a|b) 1. \end{aligned}$$

For notational convenience, we do not write the algebra unit explicitly in the rest of the paper. Thus $a \circ b = a \vee b + (a|b)$. A few more examples might be useful.

$$\begin{aligned} (a \vee b) \circ c &= a \vee b \vee c + (a|c) b + (b|c) a, \\ a \circ (b \vee c) &= a \vee b \vee c + (a|c) b + (a|b) c, \\ a \circ b \circ c &= a \vee b \vee c + (a|b) c + (a|c) b + (b|c) a, \\ a \circ b \circ c \circ d &= a \vee b \vee c \vee d + (a|b) c \vee d + (a|c) b \vee d \\ &\quad + (b|c) a \vee d + (a|d) b \vee c + (b|d) a \vee c + (c|d) a \vee b \\ &\quad + (a|b)(c|d) + (a|c)(b|d) + (b|c)(a|d), \\ (a \vee b) \circ (c \vee d) &= a \vee b \vee c \vee d + (a|c) b \vee d + (b|c) a \vee d \\ &\quad + (a|d) b \vee c + (b|d) a \vee c + (a|c)(b|d) + (b|c)(a|d). \end{aligned}$$

We shall need some simple properties of the twisted product

$$\begin{aligned} u \circ 1 &= 1 \circ u = u, \\ u \circ (v + w) &= u \circ v + u \circ w, \\ u \circ (\lambda v) &= (\lambda u) \circ v = \lambda (u \circ v). \end{aligned}$$

As an exercise, we prove that

$$(u|v) = \varepsilon(u \circ v). \tag{9}$$

We apply the counit ε to definition (8) of the twisted product:

$$\varepsilon(u \circ v) = \sum \varepsilon(u_{(1)} \vee v_{(1)})(u_{(2)}|v_{(2)}).$$

The counit is an algebra homomorphism, i.e. $\varepsilon(u_{(1)} \vee v_{(1)}) = \varepsilon(u_{(1)}) \varepsilon(v_{(1)})$. Thus, by linearity of the Laplace pairing,

$$\varepsilon(u \circ v) = \sum (\varepsilon(u_{(1)}) u_{(2)} | \varepsilon(v_{(1)}) v_{(2)}).$$

The counit property is precisely the identity $\sum \varepsilon(u_{(1)}) u_{(2)} = u$. This proves equation (9).

3.1. Proof of associativity

The twisted product is associative [5]: $u \circ (v \circ w) = (u \circ v) \circ w$. Indeed, this follows from the fact that it is a twist deformation. However, as the associativity is an important result, we provide here for the convenience of the reader an alternative self-contained proof.

We first need a useful lemma:

$$\begin{aligned} \Delta(u \circ v) &= \sum (u_{(1)} \vee v_{(1)}) \otimes (u_{(2)} \circ v_{(2)}) \qquad (10) \\ &= \sum (u_{(1)} \circ v_{(1)}) \otimes (u_{(2)} \vee v_{(2)}). \qquad (11) \end{aligned}$$

The proof is easy, we start from the definition of the twisted product (8) to write

$$\Delta(u\circ v) = \sum \Delta(u_{(1)}\vee v_{(1)})(u_{(2)}|v_{(2)}).$$

Now we use the definition (2) of the coproduct of $u_{(1)}\vee v_{(1)}$:

$$\Delta(u\circ v) = \sum(u_{(1)(1)}\vee v_{(1)(1)})\otimes(u_{(1)(2)}\vee v_{(1)(2)})(u_{(2)}|v_{(2)}).$$

The coassociativity of the coproduct of u means that any triple $(u_{(1)(1)}, u_{(1)(2)}, u_{(2)})$ can be replaced by $(u_{(1)}, u_{(2)(1)}, u_{(2)(2)})$. Therefore,

$$\Delta(u\circ v) = \sum(u_{(1)}\vee v_{(1)(1)})\otimes(u_{(2)(1)}\vee v_{(1)(2)})(u_{(2)(2)}|v_{(2)}).$$

We use now the coassociativity of the coproduct of v

$$\Delta(u\circ v) = \sum(u_{(1)}\vee v_{(1)})\otimes(u_{(2)(1)}\vee v_{(2)(1)})(u_{(2)(2)}|v_{(2)(2)})$$

and the definition of the twisted product brings

$$\Delta(u\circ v) = \sum(u_{(1)}\vee v_{(1)})\otimes(u_{(2)}\circ v_{(2)}).$$

To obtain the other identity of the lemma, we start from $u\circ v = \sum(u_{(1)}|v_{(1)})u_{(2)}\vee v_{(2)}$ or we use the cocommutativity of the coproduct.

The second lemma is

$$(u|v\circ w) = (u\circ v|w). \tag{12}$$

The proof is straightforward

$$\begin{aligned}(u|v\circ w) &= \sum(u|v_{(1)}\vee w_{(1)})(v_{(2)}|w_{(2)})\\ &= \sum(u_{(1)}|v_{(1)})(u_{(2)}|w_{(1)})(v_{(2)}|w_{(2)})\\ &= \sum(u_{(1)}|v_{(1)})(u_{(2)}\vee v_{(2)}|w) = (u\circ v|w).\end{aligned}$$

The first line is the definition of the twisted product (8), the second line is the expansion of the Laplace pairing (5), the third one is the Laplace identity (4) and the last equality is again equation (8).

The associativity of the twisted product follows immediately from these two lemmas. From definition (8) and the first lemma we obtain

$$\begin{aligned}u\circ(v\circ w) &= \sum u_{(1)}\vee(v\circ w)_{(1)}\,(u_{(2)}|(v\circ w)_{(2)})\\ &= \sum u_{(1)}\vee(v_{(1)}\vee w_{(1)})\,(u_{(2)}|v_{(2)}\circ w_{(2)}).\end{aligned}$$

From the associativity of the symmetric product and the second lemma we obtain

$$u\circ(v\circ w) = \sum u_{(1)}\vee v_{(1)}\vee w_{(1)}\,(u_{(2)}\circ v_{(2)}|w_{(2)}).$$

Now the first lemma enables us to rewrite this as

$$u\circ(v\circ w) = \sum(u\circ v)_{(1)}\vee w_{(1)}\,((u\circ v)_{(2)}|w_{(2)}) = (u\circ v)\circ w.$$

3.2. Inversion of twisted product

The fact that the twisted product is defined through a twist deformation means that it can be inverted. That is, the symmetric product can be written in terms of the twisted product. Indeed, the formula is very simple and uses the antipode (see e.g. [1, 25]:

$$u\vee v = \sum (\mathrm{s}(u_{(1)})|v_{(1)})u_{(2)}\circ v_{(2)} = \sum (u_{(1)}|\mathrm{s}(v_{(1)}))u_{(2)}\circ v_{(2)}.$$

The proof is simple:

$$\begin{aligned}\sum (\mathrm{s}(u_{(1)})|v_{(1)})u_{(2)}\circ v_{(2)} &= \sum (\mathrm{s}(u_{(1)})|v_{(1)})(u_{(2)(1)}|v_{(2)(1)})u_{(2)(2)}\vee v_{(2)(2)}\\ &= \sum (\mathrm{s}(u_{(1)(1)})|v_{(1)(1)})(u_{(1)(2)}|v_{(1)(2)})u_{(2)}\vee v_{(2)}\\ &= \sum (\mathrm{s}(u_{(1)(1)})\vee u_{(1)(2)}|v_{(1)})u_{(2)}\vee v_{(2)}\\ &= \sum \varepsilon(u_{(1)})(1|v_{(1)})u_{(2)}\vee v_{(2)}\\ &= \sum \varepsilon(u_{(1)})\varepsilon(v_{(1)})u_{(2)}\vee v_{(2)} = u\vee v.\end{aligned}$$

The last line was obtained because of (7). It is also possible to recover the Laplace pairing from the twisted product, as already shown in (9). Reference [25] also provides a sort of distributivity formula for twisted and symmetric products:

$$u\circ(v\vee w) = \sum (u_{(1)}\circ v)\vee \mathrm{s}(u_{(2)(1)})\vee(u_{(2)(2)}\circ w) \tag{13}$$

For this result we need the following lemma

$$\sum s(u_{(1)})\vee(u_{(2)}\circ v) = \sum v_{(1)}(u|v_{(2)})$$

which is proved easily

$$\begin{aligned}\sum s(u_{(1)})\vee(u_{(2)}\circ v) &= \sum s(u_{(1)})\vee u_{(2)(1)}\vee v_{(1)}(u_{(2)(2)}|v_{(2)})\\ &= \sum s(u_{(1)(1)})\vee u_{(1)(2)}\vee v_{(1)}(u_{(2)}|v_{(2)})\\ &= \sum \varepsilon(u_{(1)})v_{(1)}(u_{(2)}|v_{(2)}) = \sum v_{(1)}(u|v_{(2)}).\end{aligned}$$

Now, we can prove (13). Using the last lemma, we obtain

$$\sum (u_{(1)}\circ v)\vee \mathrm{s}(u_{(2)(1)})\vee(u_{(2)(2)}\circ w) = \sum (u_{(1)}\circ v)\vee w_{(1)}(u_{(2)}|w_{(2)}).$$

On the other hand, we have

$$\begin{aligned}\sum u\circ(v\vee w) &= \sum u_{(1)}\vee v_{(1)}\vee w_{(1)}(u_{(2)}|v_{(2)}\vee w_{(2)})\\ &= \sum u_{(1)}\vee v_{(1)}\vee w_{(1)}(u_{(2)(1)}|v_{(2)})(u_{(2)(2)}|w_{(2)})\\ &= \sum u_{(1)(1)}\vee v_{(1)}\vee w_{(1)}(u_{(1)(2)}|v_{(2)})(u_{(2)}|w_{(2)})\\ &= \sum (u_{(1)}\circ v)\vee w_{(1)}(u_{(2)}|w_{(2)}).\end{aligned}$$

This proves the lemma.

In the proofs of the lemmas of the present section and of the coassociativity of the twisted product, we never used the commutativity of the symmetric product. Therefore, all our results are still valid if we work in the tensor algebra $T(V)$ instead of the symmetric algebra $S(V)$.

3.3. Commutative twisted product

In quantum field theory, the time-ordered product is not considered as a product but as an operation $T(\varphi_1\cdots\varphi_n)$ that orders the operator product of $\varphi_1, \ldots, \varphi_n$ according to decreasing times (see, e.g. [26] p.177). Since the arguments of T commute, we can consider T as a map from $S(V)$ to $S(V)$. Moreover, we shall show that the time-ordered product can indeed be considered as a commutative twisted product. Thus, from now on in this section, the twisted product will be considered commutative. We first show that this is equivalent to the condition $(a|b)=(b|a)$ for all $a,b\in V$.

If the twisted product is commutative, then in particular $a\circ b=b\circ a$, so that $a\vee b+(a|b)=b\vee a+(b|a)$ and $(a|b)=(b|a)$. Conversely, if $(a|b)=(b|a)$ for all $a,b\in V$, then $u\circ v=v\circ u$ for all $u,v\in S(V)$. To prove the last statement, we first show that, if $(a|b)=(b|a)$ for all $a,b\in V$, then $(u|v)=(v|u)$ for all $u,v\in S(V)$. This follows immediately from the definition (6) of the permanent. Because of the commutativity of the symmetric product $\vee$ we obtain

$$u\circ v=\sum u_{(1)}\vee v_{(1)}\,(u_{(2)}|v_{(2)})=\sum v_{(1)}\vee u_{(1)}\,(v_{(2)}|u_{(2)})=v\circ u.$$

3.4. The T-map

We define recursively a linear map T from $S(V)$ to $S(V)$ by $T(1)=1$, $T(a)=a$ for a in V and $T(u\vee v)=T(u)\circ T(v)$. More explicitly, $T(a_1\vee\ldots\vee a_n)=a_1\circ\ldots\circ a_n$. The twisted product is associative and commutative, therefore T is well defined on $S(V)$. This T-map is the usual time-ordering operator of quantum field theory.

The main property we shall need in the following is

$$\Delta T(u)=\sum u_{(1)}\otimes T(u_{(2)})=\sum T(u_{(1)})\otimes u_{(2)}. \tag{14}$$

We use a recursive argument. Equation (14) is true for $u=1$ or $u=a$ and $v=1$ or $v=b$. Now

$$\Delta T(u\vee v)=\Delta\big(T(u)\circ T(v)\big).$$

By equation (10) we obtain

$$\Delta T(u\vee v)=\sum\big(T(u)_{(1)}\vee T(v)_{(1)}\big)\otimes\big(T(u)_{(2)}\circ T(v)_{(2)}\big).$$

The recursion hypothesis yields

$$\begin{aligned}\Delta T(u\vee v)&=\sum\big(u_{(1)}\vee v_{(1)}\big)\otimes\big(T(u_{(2)})\circ T(v_{(2)})\big)\\&=\sum(u\vee v)_{(1)}\otimes T(u_{(2)}\vee v_{(2)}).\end{aligned}$$

This gives us the expected result. The symmetric identity in equation (14) is obtained by cocommutativity of the coproduct.

3.5. Exponentiation

Now, we shall derive a result obtained by Anderson [27] and rediscovered several times [28]: the T-map can be written as the exponential of an operator Σ. As an application of the Laplace Hopf algebra, we give a purely algebraic proof of this. For simplicity, we consider a finite dimensional vector space V.

First, we define a derivation δ_k attached to a basis $\{e_k\}$ of the vector space V. This derivation is defined as the linear operator on $S(V)$ satisfying the following properties

$$\begin{aligned} \delta_k 1 &= 0, \\ \delta_k e_j &= \delta_{kj}, && (15) \\ \delta_k(u \vee v) &= (\delta_k u)\vee v + u\vee(\delta_k v). && (16) \end{aligned}$$

In the second equation, δ_{kj} is 1 for $j = k$ and zero otherwise. The Leibniz relation (16) gives us $\delta_1(e_1\vee e_2) = e_2$ and $\delta_2(e_1\vee e_2) = e_1$, for example. From this definition, it can be shown recursively that the derivatives commute: $\delta_i\delta_j = \delta_j\delta_i$. Note that the derivation does not act on the Laplace pairing: $\delta_i(u|v) = 0$.

Now we define the infinitesimal T-map as

$$\Sigma = \frac{1}{2}\sum_{ij}(e_i|e_j)\delta_i\delta_j,$$

where the sum is over all elements of the basis of V.

We proceed in several steps to show that $T = \exp\Sigma$. First, we show that

$$[\Sigma, a] = \sum_i (a|e_i)\delta_i. \qquad (17)$$

To do that, we apply Σ to an element $e_k\vee u$

$$\begin{aligned} \Sigma(e_k\vee u) &= \frac{1}{2}\sum_{ij}(e_i|e_j)\delta_i\delta_j(e_k\vee u) \\ &= \frac{1}{2}\sum_{ij}(e_i|e_j)\delta_i(\delta_{jk}u + e_k\vee\delta_j u) \\ &= \frac{1}{2}\sum_i(e_i|e_k)\delta_i u + \frac{1}{2}\sum_j(e_k|e_j)\delta_j u + \frac{1}{2}\sum_{ij}(e_i|e_j)e_k\vee\delta_i\delta_j u \\ &= \sum_j(e_k|e_j)\delta_j u + e_k\vee\Sigma u. \end{aligned}$$

If we extend this by linearity to V we obtain

$$\Sigma(a\vee u) = a\vee\Sigma u + \sum(a|e_j)\delta_j u. \qquad (18)$$

And more generally

$$\Sigma(u\vee v) = (\Sigma u)\vee v + u\vee(\Sigma v) + \sum_{ij}(e_i|e_j)(\delta_i u)\vee(\delta_j v).$$

From the commutation of the derivations we obtain $[\Sigma, \delta_k] = 0$ and $[\Sigma, [\Sigma, a]] = 0$. Therefore, the classical formula ([26] p.167) yields

$$\mathrm{e}^{\Sigma} a \mathrm{e}^{-\Sigma} = a + [\Sigma, a],$$

so that

$$[\mathrm{e}^{\Sigma}, a] = \sum_i (a|e_i)\delta_i \mathrm{e}^{\Sigma} = \mathrm{e}^{\Sigma}\sum_i (a|e_i)\delta_i.$$

This can be written more precisely as

$$\mathrm{e}^{\Sigma}(a\vee u) = a\vee(\mathrm{e}^{\Sigma}u) + [\Sigma, a](\mathrm{e}^{\Sigma}u). \qquad (19)$$

In the course of the proof of Wick's theorem, we shall derive equation (27) which can be rewritten

$$a \circ u = a\vee u + [\Sigma, a]u. \tag{20}$$

Now we have all we need to prove inductively that $T = \mathrm{e}^{\Sigma}$. We have $T(1) = \mathrm{e}^{\Sigma}1 = 1$ and $T(a) = a = \mathrm{e}^{\Sigma}a$ because $\Sigma a = 0$. Now assume that the property is true up to degree k, we take an element u of $S^k(V)$ and calculate

$$\begin{aligned} T(a\vee u) &= a \circ T(u) = a\vee T(u) + [\Sigma, a]T(u) \\ &= a\vee \mathrm{e}^{\Sigma}u + [\Sigma, a]\mathrm{e}^{\Sigma}u = \mathrm{e}^{\Sigma}(a\vee u), \end{aligned}$$

where we used equation (20), then equation (19). Thus the property is true for $a\vee u$ whose degree is $k+1$.

3.6. The scalar t-map

It is possible to write the T-map as a sum of scalars multiplied by elements of $S(V)$. In fact we can show that

$$T(u) = \sum t(u_{(1)})u_{(2)}, \tag{21}$$

where the map t is a linear map from $S(V)$ to $\mathbb{C}$, called the t-map and defined recursively by $t(1) = 1$, $t(a) = 0$ for $a \in V$ and

$$t(u\vee v) = \sum t(u_{(1)})t(v_{(1)})(u_{(2)}|v_{(2)}). \tag{22}$$

The proof is recursive. If the property is true up to degree k, we take $w = u\vee v$, where u and v have degree k or smaller and we calculate

$$\begin{aligned} T(w) = T(u) \circ T(v) &= \sum t(u_{(1)})t(v_{(1)})u_{(2)} \circ v_{(2)} \\ &= \sum t(u_{(1)})t(v_{(1)})(u_{(2)(1)}|v_{(2)(1)})u_{(2)(2)}\vee v_{(2)(2)} \\ &= \sum t(u_{(1)(1)})t(v_{(1)(1)})(u_{(1)(2)}|v_{(1)(2)})u_{(2)}\vee v_{(2)} \\ &= \sum t(u_{(1)}\vee v_{(1)})u_{(2)}\vee v_{(2)} = \sum t(w_{(1)})w_{(2)}, \end{aligned}$$

where the first line is the definition of $T(u\vee v)$ and the recursion hypothesis, the second line is the definition of the twisted product, the third line is the coassociativity of the coproduct and the last line is the definition of t.

Notice that the t-map is well defined because $t(u) = \varepsilon(T(u))$:

$$\varepsilon(T(u)) = \sum t(u_{(1)})\varepsilon(u_{(2)}) = t\big(\sum u_{(1)}\varepsilon(u_{(2)})\big) = t(u).$$

The scalar t-map can be calculated from its definition (22). The first degrees give

$$\begin{aligned} t(a\vee b) &= (a|b), \\ t(a\vee b\vee c\vee d) &= (a|b)(c|d) + (a|c)(b|d) + (a|d)(b|c). \end{aligned}$$

In general, $t(u) = 0$ if $|u|$ is odd and $t(u)$ is given by the following formula if $u = a_1\vee \ldots \vee a_{2n}$:

$$t(u) = \sum_{\sigma} \prod_{k=1}^{n} (a_{\sigma(k)}|a_{\sigma(k+n)}). \tag{23}$$

The sum is over the $(2n-1)!!$ permutations σ of $\{1, \ldots, 2n\}$ such that $\sigma(1) < \sigma(2) < \cdots < \sigma(n)$ and $\sigma(k) < \sigma(n+k)$ for all $k = 1, \ldots, n$. The proof is given by Caianiello

[29]. Notice that, if the t-map was a state, equation (23) would show that it is a quasifree state [30].

The deeper reason behind the T-map is Hopf algebra cohomology. If the Laplace pairing is symmetric it is an exact 2-cocycle and thus arises as the boundary of a 1-cochain. This is t. Furthermore the twist deformed algebra (with twisted product) is then isomorphic to the original one (with symmetric product) giving rise to the algebra automorphism T. The detailed explanation of these facts is deferred to [24].

4. Relation to quantum field theory

In this section, we make a connection between the Hopf algebra approach and the usual quantum field formalism. We first consider the algebra V of field operators $\phi(f)$, equipped with the normal product. Then, we define a Laplace pairing such that the twisted product reproduces the standard operator product of quantum field theory. In other words, we define the operator product as a deformation of the normal product. Textbooks sometimes proceed in the opposite direction and describe a kind of "normal ordering operation", denoted by : :, which takes a product of operators and puts all creation operators on the left and all annihilation operators on the right (see e.g. [26] p.111). However, such an operation would not be well-defined ([31] p.28). For instance, $a^\dagger(\mathbf{k})\cdot a(\mathbf{q})+\delta(\mathbf{k}-\mathbf{q})$ and $a(\mathbf{q})\cdot a^\dagger(\mathbf{k})$ are equal as operators but their normal products obtained after the action of : : are $a^\dagger(\mathbf{k})\cdot a(\mathbf{q})+\delta(\mathbf{k}-\mathbf{q})$ and $a^\dagger(\mathbf{k})\cdot a(\mathbf{q})$, which are different. Thus, it is not consistent to consider a normal product as obtained from the transformation of an operator product.

4.1. Fock space and definition of V

We first need to define a vector space V of operators, from which the space of normal products $S(V)$ will be constructed. The elements of V are defined as operators acting on a Fock space. For scalar bosons, we take $M=\mathbb{R}^N$ ($N=3$ for the usual space-time) and we consider the complex Hilbert space of square integrable functions on M: $H=L^2(M)$. To define the symmetric Fock space, we follow Ref.[32], p.53. If $H^{\otimes n}=H\otimes\cdots\otimes H$ and S_n acts on $H^{\otimes n}$ by

$$S_n\big(f_1(\mathbf{k}_1)\otimes\cdots\otimes f_n(\mathbf{k}_n)\big)=\frac{1}{n!}\sum_{\sigma\in\mathcal{S}_n} f_{\sigma(1)}(\mathbf{k}_1)\otimes\cdots\otimes f_{\sigma(n)}(\mathbf{k}_n),$$

where $\mathcal{S}_n$ is the group of permutations of n elements, then the symmetric Fock space is

$$\mathcal{F}_s(H)=\bigoplus_{n=0}^{\infty} S_n H^{\otimes n},$$

where $S_0H^{\otimes 0}=\mathbb{C}$ and $S_1H^{\otimes 1}=H$. An element $|\psi\rangle$ of $\mathcal{F}_s(H)$ can be written

$$|\psi\rangle=(\psi_0,\psi_1(\mathbf{k}_1),\ldots,\psi_n(\mathbf{k}_1,\ldots,\mathbf{k}_n),\ldots),$$

where $\psi_0\in\mathbb{C}$ and each $\psi_n(\mathbf{k}_1,\ldots,\mathbf{k}_n)\in L^2(M^n)$ is left invariant under any permutation of the variables. The components of $|\psi\rangle$ satisfy

$$||\,|\psi\rangle||^2=\langle\psi|\psi\rangle=|\psi_0|^2+\sum_{n=1}^{\infty}\int|\psi_n(\mathbf{k}_1,\ldots,\mathbf{k}_n)|^2\mathrm{d}\mathbf{k}_1\ldots\mathrm{d}\mathbf{k}_n<\infty.$$

An element of $\mathcal{F}_s(H)$ for which $\psi_n=0$ for all but finitely many n is called a finite particle vector. The set of finite particle vectors is denoted by F_0.

We define now the annihilation operator $a(\mathbf{k})$ by its action in $\mathcal{F}_s(H)$. The action of the operator $a(\mathbf{k})$ on a vector $|\psi\rangle$ is given by the coordinates $(a(\mathbf{k})|\psi\rangle)_n$ of $a(\mathbf{k})|\psi\rangle$ ([33] p.218):

$$(a(\mathbf{k})|\psi\rangle)_n(\mathbf{k}_1,\ldots,\mathbf{k}_n) = \sqrt{n+1}\,\psi_{n+1}(\mathbf{k},\mathbf{k}_1,\ldots,\mathbf{k}_n). \qquad (24)$$

The domain of $a(\mathbf{k})$ is $D_{\mathcal{S}} = \{|\psi\rangle \in F_0, \psi_n \in \mathcal{S}(M^n) \text{ for } n>0\}$, where $\mathcal{S}(M^n)$ is the Fréchet space of functions of rapid decrease* on M.

The adjoint of $a(\mathbf{k})$ is not a densely defined operator but $a^\dagger(\mathbf{k})$ is well-defined as a quadratic form on $D_{\mathcal{S}} \times D_{\mathcal{S}}$: if $|\varphi\rangle$ and $|\psi\rangle$ belong to $D_{\mathcal{S}}$ then ([33], p.219)

$$\langle\varphi|a^\dagger(\mathbf{k})\psi\rangle = \langle a(\mathbf{k})\varphi|\psi\rangle,$$

If $|\psi\rangle \in D_{\mathcal{S}}$, it can be checked that $a(\mathbf{k})|\psi\rangle \in D_{\mathcal{S}}$. Thus, we can calculate the operator product $a(\mathbf{k}_1)\cdot a(\mathbf{k}_2)|\psi\rangle \in D_{\mathcal{S}}$ (the operator product is denoted by a dot $\cdot$). The definition (24) of $a(\mathbf{k})$ and the fact that the components of $|\psi\rangle$ are symmetric enable us to prove that $a(\mathbf{k}_1)\cdot a(\mathbf{k}_2) = a(\mathbf{k}_2)\cdot a(\mathbf{k}_1)$. Therefore $a^\dagger(\mathbf{k}_1)\cdot a^\dagger(\mathbf{k}_2) = a^\dagger(\mathbf{k}_2)\cdot a^\dagger(\mathbf{k}_1)$ but nothing is known about $a(\mathbf{k}_1)\cdot a^\dagger(\mathbf{k}_2)$, because $a^\dagger(\mathbf{k}_2)$ is defined as a quadratic form, which means that all the $a^\dagger(\mathbf{k})$ must be on the left of all the $a(\mathbf{k})$.

The vectors $a^\dagger(\mathbf{k}_1)\cdot\ldots\cdot a^\dagger(\mathbf{k}_m)\cdot a(\mathbf{q}_1)\cdot\ldots\cdot a(\mathbf{q}_n)$ for $m\geq 0$ and $n\geq 0$ generate a vector space denoted by W. The element given by $n=m=0$ is 1, the unit operator. The space W is equipped with a product, called the normal product and denoted by $\vee$. If $u = a^\dagger(\mathbf{k}_1)\cdot\ldots\cdot a^\dagger(\mathbf{k}_m)\cdot a(\mathbf{q}_1)\cdot\ldots\cdot a(\mathbf{q}_n)$ and $v\in W$, the normal product of u and v is

$$u\vee v = v\vee u = a^\dagger(\mathbf{k}_1)\cdot\ldots\cdot a^\dagger(\mathbf{k}_m)\cdot v\cdot a(\mathbf{q}_1)\cdot\ldots\cdot a(\mathbf{q}_n).$$

This product is well defined because the creation operators commute as well as the annihilation operators. W is a commutative and associative $*$-algebra with unit, generated by $a(\mathbf{k})$ and $a^\dagger(\mathbf{k})$. The $*$-structure is defined by $a(\mathbf{k})^* = a^\dagger(\mathbf{k})$.

Now, we are ready to define the vector space V. A smoothed field operator is an operator $\phi(f)$ defined by

$$\phi(f) = \int \mathrm{d}x \int \frac{\mathrm{d}\mathbf{k}}{(2\pi)^3\sqrt{2\omega_k}}\Big(f(x)\mathrm{e}^{-ip\cdot x}a(\mathbf{k}) + f(x)\mathrm{e}^{ip\cdot x}a^\dagger(\mathbf{k})\Big),$$

where $\omega_k = \sqrt{m^2+|\mathbf{k}|^2}$ and $p=(\omega_k,\mathbf{k})$. The vector space V is

$$V = \{\phi(f), f\in\mathcal{D}(\mathbb{R}^{N+1})\},$$

where $\mathcal{D}(\mathbb{R}^{N+1})$ is the space of infinitely differentiable functions with compact support in $\mathbb{R}^{N+1}$. V is indeed a vector space, with $\phi(f)+\phi(g) = \phi(f+g)$ and $\lambda\phi(f) = \phi(\lambda f)$. These smoothed fields can also be written $\phi(f) = \int \mathrm{d}x f(x)\phi(x)$, where

$$\phi(x) = \int \frac{\mathrm{d}\mathbf{k}}{(2\pi)^3\sqrt{2\omega_k}}\Big(\mathrm{e}^{-ip\cdot x}a(\mathbf{k}) + \mathrm{e}^{ip\cdot x}a^\dagger(\mathbf{k})\Big).$$

The field operators $\phi(x)$ are defined as quadratic forms (see [33], p. 223). They look like a basis of V but $\phi(x)$ does not belong to V.

* We recall that a function of rapid decrease is an infinitely differentiable complex-valued function such that

$$||f||_{\alpha,\beta} = \sup_{x\in M} |x^\alpha D^\beta f(x)| < \infty,$$

for all $\alpha,\beta\in I_+^N$, where I_+^N is the set of all N-tuples of nonnegative integers (see Ref. [32], p.133).

4.2. The Hopf algebra $S(V)$ of normal products

Now that we have the vector space V, the Hopf algebra $S(V)$ is defined according to Section 2.1. However, we want to identify elements of $S(V)$ (and not only V) as operators in the sense of Section 4.1. We know that $S^0(V)=\mathbb{C}1$, where 1 is the unit operator: for any $|\psi\rangle \in F_s(H)$, $1|\psi\rangle = |\psi\rangle$. We also know that $S^1(V)=V$. To identify $S^n(V)$ as a space of operators in the desired sense we only need to deduce the symmetric product of $S^n(V)$ from the normal product of creation and annihilation operators. For example

$$\begin{aligned}
\phi(f)\vee\phi(g) &= \int \mathrm{d}x\mathrm{d}x' \int \frac{\mathrm{d}\mathbf{k}}{(2\pi)^3\sqrt{2\omega_k}} \frac{\mathrm{d}\mathbf{k}'}{(2\pi)^3\sqrt{2\omega_{k'}}} f(x)g(x') \\
&\quad \times \Big(\mathrm{e}^{-i(p\cdot x+p'\cdot x')}a(\mathbf{k})\vee a(\mathbf{k}') + \mathrm{e}^{-i(p\cdot x-p'\cdot x')}a(\mathbf{k})\vee a^\dagger(\mathbf{k}') \\
&\quad + \mathrm{e}^{i(p\cdot x-p'\cdot x')}a^\dagger(\mathbf{k})\vee a(\mathbf{k}') + \mathrm{e}^{i(p\cdot x+p'\cdot x')}a^\dagger(\mathbf{k})\vee a^\dagger(\mathbf{k}')\Big) \\
&= \int \mathrm{d}x\mathrm{d}x' \int \frac{\mathrm{d}\mathbf{k}}{(2\pi)^3\sqrt{2\omega_k}} \frac{\mathrm{d}\mathbf{k}'}{(2\pi)^3\sqrt{2\omega_{k'}}} f(x)g(x') \\
&\quad \times \Big(\mathrm{e}^{-i(p\cdot x+p'\cdot x')}a(\mathbf{k})\cdot a(\mathbf{k}') + \mathrm{e}^{-i(p\cdot x-p'\cdot x')}a^\dagger(\mathbf{k}')\cdot a(\mathbf{k}) \\
&\quad + \mathrm{e}^{i(p\cdot x-p'\cdot x')}a^\dagger(\mathbf{k})\cdot a(\mathbf{k}') + \mathrm{e}^{i(p\cdot x+p'\cdot x')}a^\dagger(\mathbf{k})\cdot a^\dagger(\mathbf{k}')\Big).
\end{aligned}$$

In the above expression, the integral over x and x' gives the Fourier transform of f and g, which is square integrable because of Plancherel's theorem ([33] p.4). Thus, theorem X.44 of Ref. [33] p.220 can be applied to show that the product is a well defined operator on $D_{\mathcal{S}}$. The same reasoning shows that the symmetric product $\vee$ is well defined in $S(V)$. By definition the coproduct of $S(V)$ is generated by

$$\Delta\phi(f) = \phi(f)\otimes 1 + 1\otimes\phi(f).$$

The antipode is defined by $\mathrm{s}(\phi(f_1)\vee\ldots\vee\phi(f_n)) = (-1)^n\phi(f_1)\vee\ldots\vee\phi(f_n)$.

The counit has a direct quantum field meaning: it is the expectation value over the vacuum. We first define a vector of $D_{\mathcal{S}}$ called the vacuum: $|0\rangle = (1,0,0,\ldots)$. From the definition (24) of $a(\mathbf{k})$ we see that $a(\mathbf{k})|0\rangle = 0$ and $\langle 0|a^\dagger(\mathbf{k}) = 0$. Therefore, $\langle 0|u|0\rangle = 0 = \varepsilon(u)$, for any $u\in S^n(V)$ if $n>0$. Moreover, $\langle 0|1|0\rangle = \langle 0|0\rangle = 1 = \varepsilon(1)$. Thus, the map that sends $u\in S(V)$ to $\langle 0|u|0\rangle$ is a linear map from $S(V)$ to $\mathbb{C}$ which is equal to ε on $S^n(V)$ for all n. Therefore, $\varepsilon(u)=\langle 0|u|0\rangle$. This identity is an additional argument in favor of the fact that Hopf algebras are a natural framework for quantum field theory.

The Hopf algebra $S(V)$ has a $*$-structure (see appendix) defined by $(\lambda 1)^* = \bar\lambda 1$ for $\lambda\in\mathbb{C}$, $\phi(g)^* = \phi(\bar g)$ for $a(g)\in V$, and $(u\vee v)^* = v^*\vee u^* = u^*\vee v^*$. This makes $S(V)$ a Hopf $*$-algebra because $*$ is an antilinear involution and a homomorphism of coalgebra:

$$\begin{aligned}
\Delta(\phi(g)^*) &= \Delta\phi(\bar g) = \phi(\bar g)\otimes 1 + 1\otimes\phi(\bar g) \\
&= \phi(g)^*\otimes 1 + 1\otimes\phi(g)^* = (\Delta\phi(g))^*,
\end{aligned}$$

so that $\Delta u^* = \sum {u_{(1)}}^*\otimes {u_{(2)}}^*$ for any $u\in S(V)$. Moreover, the $*$-structure respects the grading, because $u^*\in S^n(V)$ if $u\in S^n(V)$ and the condition on the antipode is satisfied ([11] p.87): for $u\in S^n(V)$ we have $S(u) = (-1)^n u$ and $S(u^*) = (-1)^n u^*$, thus

$$S\big(S(u)^*\big)^* = (-1)^n S(u^*)^* = (u^*)^* = u.$$

Thus, $S(V)$ is a Hopf $*$-algebra.

4.3. Alternative constructions

Our definition of $S(V)$ as operators acting on a Fock space is the most standard one but other methods exist to define $S(V)$. For instance, we can consider an abstract Borchers-Uhlmann algebra [34, 35, 36] equipped with a state. The GNS construction then yields a representation of the algebra as operators acting on a Hilbert space. The Borchers-Uhlmann approach is interesting because it can be used in a curved space-time.

4.4. Green functions and propagators

The Laplace pairings corresponding to operator product and time-ordered product are to be defined from various Green functions and propagators. As many conflicting conventions can be found for these in the literature, we give now explicit expressions for the functions we use in the case of the free scalar field. We consider a flat four-dimensional space-time with diagonal metric $(1,-1,-1,-1)$. The following formulas are compiled from Refs.[26], p. 33 and p.133, [37], Sections 5 and 13, [38], Section 2.3 and [39] Section 2-1, after conversion to our notation.

The positive frequency function is

$$G_+(x) = \int \frac{d\mathbf{k}}{(2\pi)^3 2\omega_k} \mathrm{e}^{-ip\cdot x} = \int \frac{d^4p}{(2\pi)^3} \delta(p^2-m^2)\theta(p^0)\mathrm{e}^{-ip\cdot x},$$

where $\omega_k = \sqrt{m^2+|\mathbf{k}|^2}$ and $p=(\omega_k,\mathbf{k})$. The symmetric function is

$$\begin{aligned} G^1(x) &= (1/2)\big(G_+(x)+G_+(-x)\big) = \int \frac{d\mathbf{k}}{(2\pi)^3 2\omega_k}\cos(p\cdot x) \\ &= \int \frac{d^4p}{2(2\pi)^3}\delta(p^2-m^2)\mathrm{e}^{-ip\cdot x}. \end{aligned}$$

The commutator is

$$\begin{aligned} G(x) &= -i\big(G_+(x)-G_+(-x)\big) = -\int \frac{d\mathbf{k}}{(2\pi)^3\omega_k}\sin(p\cdot x) \\ &= -i\int \frac{d^4p}{(2\pi)^3}\delta(p^2-m^2)\mathrm{sign}(p^0)\mathrm{e}^{-ip\cdot x}. \end{aligned}$$

The commutator is related to the retarded Green function by $G_{\mathrm{ret}}(x) = -\theta(x^0)G(x)$ and to the advanced one by $G_{\mathrm{adv}}(x) = \theta(-x^0)G(x)$. Finally, the Stueckelberg-Feynman Green function is

$$G_F(x) = iG_+(x) + G_{\mathrm{adv}}(x) = -\int \frac{d^4p}{(2\pi)^4}\frac{\mathrm{e}^{-ip\cdot x}}{p^2-m^2+i\epsilon}.$$

The functions $G_{\mathrm{adv}}(x-y)$, $G_{\mathrm{ret}}(x-y)$ and $G_F(x-y)$ satisfy $(\partial_\mu\partial^\mu+m^2)G_X = \delta(x-y)$, the functions $G_+(x-y)$, $G(x-y)$ and $G^1(x-y)$ satisfy $(\partial_\mu\partial^\mu+m^2)G_X=0$. In the case of a massless scalar field, these functions are

$$\begin{aligned} G_+(x) &= -\frac{1}{4\pi^2}\big(P\frac{1}{x^2} + i\pi\,\mathrm{sign}(x^0)\delta(x^2)\big), \\ G^1(x) &= -\frac{1}{4\pi^2}P\frac{1}{x^2}, \quad G(x) = -\frac{1}{2\pi}\mathrm{sign}(x^0)\delta(x^2), \\ G_{\mathrm{ret}}(x) &= \frac{1}{2\pi}\theta(x^0)\delta(x^2), \quad G_{\mathrm{adv}}(x) = \frac{1}{2\pi}\theta(-x^0)\delta(x^2), \\ G_F(x) &= -\frac{i}{4\pi^2}\frac{1}{x^2-i\epsilon}. \end{aligned}$$

The corresponding Green functions and propagators for a real scalar field in curved space-time or in an external potential can be found in Ref.[40]. Notice, that $G_+(x)$ satisfies the Hadamard condition (see, e.g. equation (3) in [41]) because it can be written as

$$G_+(x) = \frac{1}{(2\pi)^2} \frac{1}{-x^2 + 2i\varepsilon x^0 + \epsilon^2}.$$

4.5. Operator product

In this section, we show that the choice of a Laplace pairing enables us to define a twisted product that reproduces the usual operator product of quantum field theory.

4.5.1. Laplace pairing The twisted product is defined from the Laplace pairing

$$(\varphi(f)|\varphi(g))_+ = \int \mathrm{d}x\mathrm{d}y f(x) G_+(x-y) g(y). \tag{25}$$

by (8). We denote it by $\bullet$. For example

$$\varphi(f) \bullet \varphi(g) = \varphi(f) \vee \varphi(g) + (\varphi(f)|\varphi(g))_+.$$

On the other hand, Wick's theorem gives the operator product as the sum of the normal product plus the average value over the vacuum:

$$\varphi(f) \cdot \varphi(g) = \varphi(f) \vee \varphi(g) + \langle 0|\varphi(f) \cdot \varphi(g)|0\rangle.$$

The last term can be written $\langle 0|\varphi(f) \cdot \varphi(g)|0\rangle = \int \mathrm{d}x\mathrm{d}y f(x) g(y) \langle 0|\varphi(x) \cdot \varphi(y)|0\rangle$, where $\langle 0|\varphi(x) \cdot \varphi(y)|0\rangle$ is the Wightman function $G_+(x-y)$. Thus, equation (25) shows that $\varphi(f) \bullet \varphi(g) = \varphi(f) \cdot \varphi(g)$ and the twisted product of two fields is equal to their operator product.

4.5.2. Wick's theorem We have shown that the operator product of two field operators can be defined as a twisted product. We must show now that this is still true for the product of any number of field operators. As the relation between operator product and normal product is governed by Wick's theorem, it is enough to show that Wick's theorem is satisfied in $S(V)$. For a related approach in the fermionic case, see Reference [7].

Wick's theorem is very well known, so we recall it only briefly. It states that the operator product of a given number of elements of V is equal to the sum over all possible pairs of contractions (see e.g. [42] p.209, [43] p.261, [31] p.85). A contraction $a^\bullet b^\bullet$ is the difference between the operator product and the normal product. In Wick's notation [3] $a^\bullet b^\bullet = ab - {:}ab{:}$. This pairing was used by Houriet and Kind even before Wick's article [2]. The contraction is a scalar. To express it in our notation, we identify the normal product ${:}ab{:}$ with the symmetric product $a \vee b$.

Wick's theorem is proved recursively from the following identity [3]:

$${:}a_1 \ldots a_n{:}\, b = {:}a_1 \ldots a_n b{:} + \sum_{j=1}^{n} a_j^\bullet b^\bullet \, {:}a_1 \ldots a_{j-1} a_{j+1} \ldots a_n{:} \tag{26}$$

Thus, we must show that the above identity is valid for the twisted product. In our notation, (defining $u = a_1 \vee \ldots \vee a_n$) we must prove

$$u \bullet b = u \vee b + \sum_{j=1}^{n} (a_j|b)\, a_1 \vee \ldots \vee a_{j-1} \vee a_{j+1} \vee \ldots \vee a_n. \tag{27}$$

To show this, we use the definition (8) of the twisted product and equation (1) to find

$$u \bullet b = u \vee b + \sum (u_{(1)}|b) u_{(2)}. \tag{28}$$

The Laplace pairing $(u_{(1)}|b)$ is zero if the degree of $u_{(1)}$ is different from 1. In other words, $u_{(1)}$ must be an element of V. According to the general equation (3) for Δu, this happens only for the $(1, n-1)$-shuffles. By definition, a $(1, n-1)$-shuffle is a permutation σ of $(1, \ldots, n)$ such that $\sigma(2) < \ldots < \sigma(n)$, and the corresponding terms in the coproduct of Δu are

$$\sum_{j=1}^{n} a_j \otimes a_1 \vee \ldots \vee a_{j-1} \vee a_{j+1} \vee \ldots \vee a_n.$$

Thus, we recover (27) from the Laplace identity (28). Equation (27) is equivalent to Wick's identity (26) because the symmetric product of equation (27) is the normal product of equation (26), and the Laplace pairing of equation (27) is the same as the contraction of equation (26). Hence, the left-hand sides are identical and the twisted product of u and b is the operator product of u and b. From the associativity of the operator and twisted products, we deduce that the twisted product of any number of operators is equal to their operator product. Therefore, the twisted product is equal to the operator product on $S(V)$.

Wick's theorem enables us to calculate the operator product of elements of V. The quantum group approach enables us to express the operator product of elements of $S(V)$ as

$$u_1 \bullet \ldots \bullet u_n = \Big(\prod_{i=1}^{n-1} \prod_{j=i+1}^{n} (u_{i(j-1)}|u_{j(i)}) \Big) u_{1(n)} \vee \ldots \vee u_{n(n)}, \tag{29}$$

where $u_{i(j)}$ is obtained by composing $n-1$ times the coproduct of u_i. For instance, for $n = 4$,

$$(\mathrm{Id} \otimes \mathrm{Id} \otimes \Delta)(\mathrm{Id} \otimes \Delta)\Delta u = \sum u_{(1)} \otimes u_{(2)} \otimes u_{(3)} \otimes u_{(4)}.$$

A simple example of this generalized Wick's theorem is

$$u_1 \bullet u_2 \bullet u_3 = \sum (u_{1(1)}|u_{2(1)})(u_{1(2)}|u_{3(1)})(u_{2(2)}|u_{3(2)}) u_{1(3)} \vee u_{2(3)} \vee u_{3(3)}.$$

A proof of equation (29) for the general case of $S(C)$, where C is a cocommutative coalgebra is given in [44].

4.6. Time-ordered product

In this section, we show that the choice of a Laplace pairing enables us to define a twisted product that reproduces the usual time-ordered product of quantum field theory.

We start with the Laplace pairing

$$(\varphi(f)|\varphi(g))_F = -i \int \mathrm{d}x \mathrm{d}y f(x) G_F(x-y) g(y). \tag{30}$$

We denote by $\circ$ the corresponding twisted product. For example

$$\varphi(f) \circ \varphi(g) = \varphi(f) \vee \varphi(g) + (\varphi(f)|\varphi(g))_F.$$

In particular, it can be checked that

$$\langle 0|\varphi(f) \circ \varphi(g)|0\rangle = (\varphi(f)|\varphi(g))_F = -i \int \mathrm{d}x \mathrm{d}y f(x) G_F(x-y) g(y).$$

As in section 4.5.1, we see that the twisted product of two operators is equal to their time-ordered product because $-iG_F(x-y) = \langle 0|T(\varphi(x)\varphi(y))|0\rangle$ ([33] p.214). The fact that the time-ordered product of any number of field operators is equal to their twisted product with Laplace pairing (30) follows again from Wick's theorem and is proved as for the operator product. Notice that $G_F(x)$ is a symmetric distribution (i.e. $G_F(-x) = G_F(x)$), thus the Laplace coupling $(|)_F$ is symmetric and the time-ordered product is commutative. It might be useful to stress here that the twisted product is always associative. This implies that the time-ordered product is an associative and commutative product.

The role of the counit as the vacuum expectation value thus enables us to express (smeared) correlation functions of free scalar quantum field theory. For example, the n-point function smeared with test functions $f_1, \ldots, f_n$ is given by

$$\begin{aligned}
&\langle 0|T(\varphi(f_1)\varphi(f_2)\cdots\varphi(f_n))|0\rangle \\
&= \varepsilon\left(\varphi(f_1)\circ\varphi(f_2)\circ\cdots\circ\varphi(f_n)\right) \\
&= t\left(\varphi(f_1)\vee\varphi(f_2)\vee\cdots\vee\varphi(f_n)\right),
\end{aligned}$$

where t is defined from $\circ$ as in Section 3.6. The time-ordered product is particularly important when we consider interacting quantum fields because it enters the definition of the S-matrix and the interacting Green functions. In a forthcoming publication, it will be shown that quantum groups methods allow for an easy handling of the S-matrix and provide short algebraic proofs of the linked-cluster and coupled-cluster theorems.

5. Conclusion

The quantum group approach to quantum field theory provides a second quantization without commutators. To this end the algebra of field operators with normal product is equipped with a Hopf algebra structure. Green functions are encoded in the Laplace pairing. Both operator product and time-ordered product arise as twisted products from Laplace pairings via Hopf algebra cohomology theory. The relationship between normal, operator and time-ordered product is thus completely encoded in the algebraic expressions. In particular, this is also true for Wick's theorem. Combinatorial structures and proofs in quantum field theory are considerably simplified by casting them in closed algebraic expressions.

On the other hand, the approach advocated here brings second quantization much closer in spirit to deformation quantization. If we think of the field operators with normal product as classical observables (arising from classical field configurations as in the path integral picture) their operator product as quantum observables arises indeed as a twist deformation of the classical commutative product [45, 46, 47].

Although we have considered here only free fields, the formalism extends naturally to the interacting case [8, 44]. If we consider the exterior algebra $\Lambda(V)$ instead of the symmetric algebra $S(V)$ we can carry out the second quantization of fermion fields [24].

While in this paper the Laplace pairing was considered to be scalar, it is also possible to give it a value in an algebra. This enables us to formally second-quantize associative algebras or to consider matrix-valued states, which should be useful in quantum field calculations involving degenerate vacua. Moreover, the quantum group approach provides powerful algorithms to calculate quantum field theory expressions with correlated vacua, which are used in solid-state physics and quantum chemistry [8, 48].

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Appendix

In this appendix, we make a short presentation of Hopf algebras.

A *Hopf algebra* is an algebra $\mathcal{A}$ equipped with a coassociative coproduct Δ, a counit ε and an antipode s, such that Δ and ε are algebra homomorphisms. We now introduce these concepts step by step.

First, an *algebra* is a vector space $\mathcal{A}$ over $\mathbb{C}$ equipped with an associative linear product over $\mathcal{A}$, denoted $\cdot$ and a unit, denoted 1. A product is *linear* if for any $a, b, c \in \mathcal{A}$: $a\cdot(b+c) = a\cdot b + a\cdot c$, $(a+b)\cdot c = a\cdot c + b\cdot c$ and $a\cdot(\lambda b) = \lambda(a\cdot b) = (\lambda a)\cdot b$. It is *associative* if for any $a, b, c \in \mathcal{A}$: $(a\cdot b)\cdot c = a\cdot(b\cdot c)$. The *unit* is the element of $\mathcal{A}$ such that, for any $a \in \mathcal{A}$, we have $1\cdot a = a\cdot 1 = a$. A good example of an algebra is the symmetric algebra $S(V)$ described in section 2.1.

The more unusual concept in a Hopf algebra is the *coproduct*, which is a linear map from $\mathcal{A}$ to $\mathcal{A}\otimes\mathcal{A}$, denoted by Δ. A coproduct can often be considered as giving all the ways to split an element of $\mathcal{A}$ into two "parts". For the example of the symmetric algebra $S(V)$, the coproduct of $x_1\vee\ldots\vee x_n$ (with all x_i in V) is defined as

$$\Delta x_1\vee\ldots\vee x_n = \sum_I P(I)\otimes P(I^c),$$

where the sum is over all subsets I of $\{x_1,\ldots,x_n\}$, $I^c = \{x_1,\ldots,x_n\}\backslash I$ and $P(I)$ is the symmetric product of the elements of I. For instance,

$$\begin{aligned}
\Delta 1 &= 1\otimes 1,\\
\Delta x &= x\otimes 1 + 1\otimes x,\\
\Delta x\vee y &= x\vee y\otimes 1 + x\otimes y + y\otimes x + 1\otimes x\vee y,\\
\Delta x\vee y\vee z &= x\vee y\vee z\otimes 1 + x\otimes y\vee z + y\otimes x\vee z + z\otimes x\vee y\\
&\quad + x\vee y\otimes z + x\vee z\otimes y + y\vee z\otimes x + 1\otimes x\vee y\vee z.
\end{aligned}$$

To recall this 'splitting', the action of the coproduct is denoted by $\Delta a = \sum a_{(1)}\otimes a_{(2)}$, where $a_{(1)}$ and $a_{(2)}$ are the first and second "parts" of a. For instance, if $a = x$, we have $a_{(1)} = x$ and $a_{(2)} = 1$ for the first term of Δa and $a_{(1)} = 1$ and $a_{(2)} = x$ for its second term. This is called Sweedler's notation.

In a Hopf algebra, the coproduct is compatible with the product, in other words $\Delta(a\cdot b) = (\Delta a)\cdot(\Delta b)$. Or, more precisely $\Delta(a\cdot b) = \sum(a_{(1)}\cdot b_{(1)})\otimes(a_{(2)}\cdot b_{(2)})$. In mathematical terms, we say that the coproduct is an *algebra homomorphism*.

This compatibility is often used to define the coproduct. For instance, we can consider the algebra $\mathcal{J}$ generated by the angular momentum operators J_x, J_y and J_z

(i.e. the universal enveloping algebra of su_2). The product $\cdot$ is the operator product and the coproduct is defined by the action of the angular momentum operators on products of two states $|\psi_1\rangle \otimes |\psi_2\rangle$

$$\Delta J_i = J_i \otimes 1 + 1 \otimes J_i. \tag{A.1}$$

We recognize here the coupling of angular momenta. This gives us a physical meaning of the coproduct: it is a way to go from a one-particle operator to a two-particle operator. For products of operators $J_{i_1} \cdot \ldots \cdot J_{i_n}$ the coproduct is defined recursively by its compatibility with the operator product:

$$\Delta(J_{i_1} \cdot J_{i_2} \cdot \ldots \cdot J_{i_n}) = \Delta(J_{i_1}) \cdot \Delta(J_{i_2} \cdot \ldots \cdot J_{i_n}).$$

For example $\Delta(J_x \cdot J_y) = J_x \cdot J_y \otimes 1 + J_x \otimes J_y + J_y \otimes J_x + 1 \otimes J_x \cdot J_y$. Now it is natural to go from a two-particle operator to a three-particle operator by repeating the action of the coproduct. However, since $\Delta a = \sum a_{(1)} \otimes a_{(2)}$ it is not clear on which side the second Δ should act: on $a_{(1)}$ or on $a_{(2)}$? If Δ is applied on $a_{(1)}$ we obtain

$$(\Delta \otimes \mathrm{Id})\Delta a = \sum (\Delta a_{(1)}) \otimes a_{(2)} = \sum a_{(1)(1)} \otimes a_{(1)(2)} \otimes a_{(2)},$$

where we have written $\Delta a_{(1)} = \sum a_{(1)(1)} \otimes a_{(1)(2)}$. If Δ is applied on $a_{(2)}$ we obtain

$$(\mathrm{Id} \otimes \Delta)\Delta a = \sum a_{(1)} \otimes (\Delta a_{(2)}) = \sum a_{(1)} \otimes a_{(2)(1)} \otimes a_{(2)(2)}.$$

A very important property of Hopf algebras is that the result does not depend on which side you apply the coproduct. In other word, for a Hopf algebra we require that the coproduct satisfies $(\Delta \otimes \mathrm{Id})\Delta a = (\mathrm{Id} \otimes \Delta)\Delta a$, i.e.

$$\sum a_{(1)(1)} \otimes a_{(1)(2)} \otimes a_{(2)} = \sum a_{(1)} \otimes a_{(2)(1)} \otimes a_{(2)(2)}.$$

This property is called the *coassociativity* of the coproduct. By applying again the coproduct, we can define the action of the angular momentum operators on many-particle states. The coassociativity ensures that the result does not depend on the order used. If the coproduct is seen from the splitting point of view, coassociativity means that the splitting of an element into three parts does not depend on whether you further split the first or the second part. It can be checked that the coproduct of the symmetric algebra $S(V)$ and of the universal enveloping algebra $\mathcal{J}$ are coassociative.

We still need two ingredients to make a Hopf algebra: a *counit* and an *antipode*. A *counit* is a linear map from $\mathcal{A}$ to $\mathbb{C}$, denoted by ε, such that $\sum a_{(1)}\varepsilon(a_{(2)}) = \sum \varepsilon(a_{(1)})a_{(2)} = a$. In a Hopf algebra, the counit ε is an algebra homomorphism (i.e. $\varepsilon(a \cdot b) = \varepsilon(a)\varepsilon(b)$). In most cases, the counit is very simple. For $S(V)$ we have $\varepsilon(1) = 1$, $\varepsilon(x_1 \vee \ldots \vee x_n) = 0$ for $n > 0$, for $\mathcal{J}$ we have $\varepsilon(1) = 1$, $\varepsilon(J_{i_1} \cdot \ldots \cdot J_{i_n}) = 0$ for $n > 0$. Finally the *antipode* is a linear map from $\mathcal{A}$ to $\mathcal{A}$ such that $\sum a_{(1)} \cdot \mathrm{s}(a_{(2)}) = \sum \mathrm{s}(a_{(1)}) \cdot a_{(2)} = \varepsilon(a)1$. The antipode is a kind of inverse. It is anti-multiplicative, i.e. $\mathrm{s}(a \cdot b) = \mathrm{s}(b) \cdot \mathrm{s}(a)$. In our examples: for $S(V)$ the antipode is defined by $\mathrm{s}(1) = 1$ and $\mathrm{s}(x) = -x$, so that $\mathrm{s}(x_1 \vee \ldots \vee x_n) = (-1)^n x_1 \vee \ldots \vee x_n$ for $n > 0$, for $\mathcal{J}$ the antipode is defined by $\mathrm{s}(1) = 1$ and $\mathrm{s}(J_i) = -J_i$, so that $\mathrm{s}(J_{i_1} \cdot \ldots \cdot J_{i_n}) = (-1)^n J_{i_n} \cdot \ldots \cdot J_{i_1}$ for $n > 0$.

A Hopf algebra is *commutative* if, for any $a, b \in \mathcal{A}$: $a \cdot b = b \cdot a$. A Hopf algebra is *cocommutative* if, for any $a \in \mathcal{A}$: $\sum a_{(1)} \otimes a_{(2)} = \sum a_{(2)} \otimes a_{(1)}$, in other words, if a can be split into $a_{(1)}$ and $a_{(2)}$, then there is also a splitting of a where $a_{(2)}$ is the first part and $a_{(1)}$ is the second part. The reader can check that $S(V)$ and $\mathcal{J}$ are cocommutative Hopf algebras. More generally, the universal enveloping algebra of any Lie algebra is a cocommutative Hopf algebra.

The next concept we shall use is that of a graded Hopf algebra. An algebra is *graded* if there exist subspaces $(\mathcal{A}_i)_{i\in\mathbb{N}}$ such that $\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1 \oplus \mathcal{A}_2 \oplus \cdots$ and $\mathcal{A}_m \cdot \mathcal{A}_n \subset \mathcal{A}_{m+n}$ for all $m, n \in \mathbb{N}$ (thus $1 \in \mathcal{A}_0$). An element a of $\mathcal{A}$ is said to be *homogeneous* if there is an m such that $a \in \mathcal{A}_m$. The *degree* of a is then $|a| = m$. A *graded Hopf algebra* is a Hopf algebra which is graded as an algebra and such that $\Delta\mathcal{A}_n \subset \bigoplus_{m=0}^{n} \mathcal{A}_m \otimes \mathcal{A}_{n-m}$ and $\mathrm{s}(\mathcal{A}_n) \subset \mathcal{A}_n$ for all n. If all elements of $\mathcal{A}_0$ can be written $\lambda 1$, the algebra is said to be *connected*.

In our two examples, $S(V)$ is a connected graded Hopf algebra, where the degree is given by $|x_1 \vee \ldots \vee x_n| = n$ but $\mathcal{J}$ is not a (non-trivially) graded algebra because of the commutator identity $J_x \cdot J_y - J_y \cdot J_x = iJ_z$.

Finally, we define the concept of a Hopf $*$-algebra ([11] p.86 and [1] p.31). A $*$-algebra is an algebra $\mathcal{A}$ together with a map $*$ from $\mathcal{A}$ to $\mathcal{A}$ which is antilinear ($(a+b)^* = a^* + b^*$ and $(\lambda a)^* = \bar{\lambda} a^*$, where $\bar{\lambda}$ is the complex conjugate of λ) and obeys $(a^*)^* = a$ and $(ab)^* = b^* a^*$ for any a and b in $\mathcal{A}$. A Hopf $*$-algebra $\mathcal{A}$ is a Hopf algebra which is also a $*$-algebra and which satisfies the following compatibility conditions: $\Delta(a^*) = (\Delta a)^*$ (where we define $(a \otimes b)^* = a^* \otimes b^*$), $\varepsilon(a^*) = \overline{\varepsilon(a)}$ and $S(S(a)^*)^* = a$ for any $a \in \mathcal{A}$. For more information on Hopf algebras and quantum groups, see Refs.[1, 11, 49, 50, 10].

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