Quantum groups and interacting quantum fields

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Abstract. If C is a cocommutative coalgebra, a bialgebra structure can be given to the symmetric algebra $S(C)$. The symmetric product is twisted by a Laplace pairing and the twisted product of any number of elements of $S(C)$ is calculated explicitly. This is used to recover important identities in the quantum field theory of interacting scalar bosons.

1. Introduction

Quantum groups appear to be a powerful tool for quantum field calculations. First, Fauser pointed out a connection between Wick's theorem and the concept of Laplace pairing introduced by Rota and his school [1]. In reference [2], we showed that the Laplace pairing is a quantum group concept and we solved quantum field problems with quantum group tools. The quantum group approach to free scalar fields was presented in detail in [3]. Here we show that quantum groups can also deal with interacting fields. In the first part, a bialgebra $S(C)$ is built from a cocommutative coalgebra C, the symmetric product of $S(C)$ is twisted by a Laplace pairing and general identities are derived. In the second part, these identities are translated into the language of quantum field theory.

2. The abstract setting

If C is a cocommutative coalgebra with coproduct Δ' and counit ε', the symmetric algebra $S(C)=\bigoplus_{n=0}^{\infty} S^n(C)$ can be equipped with the structure of a bialgebra. The product of the bialgebra $S(C)$ is the symmetric product (denoted by $\cdot$) and its coproduct Δ is defined on $S^1(C)=C$ by $\Delta a=\Delta' a$ and extended to $S(C)$ by $\Delta 1=1\otimes 1$ and $\Delta(u\cdot v)=\sum u_{(1)}\cdot v_{(1)}\otimes u_{(2)}\cdot v_{(2)}$. The elements of $S^n(C)$ are said to be of degree n. The counit ε of $S(C)$ is defined to be equal to ε' on $S^1(C)=C$ and extended to $S(C)$ by $\varepsilon(1)=1$ and $\varepsilon(u\cdot v)=\varepsilon(u)\varepsilon(v)$. It can be checked recursively that Δ is coassociative and cocommutative and that $\sum\varepsilon(u_{(1)})u_{(2)}=\sum u_{(1)}\varepsilon(u_{(2)})=u$. Thus, $S(C)$ is a commutative and cocommutative bialgebra which is graded as an algebra.

A Laplace pairing is a bilinear map $(|)$ from $S(C)\times S(C)$ to the complex numbers such that $(1|u)=(u|1)=\varepsilon(u)$, $(u\cdot v|w)=\sum(u|w_{(1)})(v|w_{(2)})$ and $(u|v\cdot w)=\sum(u_{(1)}|v)(u_{(2)}|w)$ for any u, v and w in $S(C)$. The powers Δ^k of the coproduct are defined by $\Delta^0 a=a$, $\Delta^1 a=\Delta a$ and $\Delta^{k+1}a=(\mathrm{Id}\otimes\ldots\otimes\mathrm{Id}\otimes\Delta)\Delta^k a$. Their action is denoted by $\Delta^k a=\sum a_{(1)}\otimes\ldots\otimes a_{(k+1)}$.

From the definition of the Laplace pairing and of the powers of the coproduct a straighforward recursive proof yields, for u^i and v^j in $S(C)$

$$(u^1\cdot\ldots\cdot u^k|v^1\cdot\ldots\cdot v^l)=\sum\prod_{i=1}^{k}\prod_{j=1}^{l}(u^i_{(j)}|v^j_{(i)}). \tag{1}$$

For example $(u\cdot v\cdot w|s\cdot t)=\sum(u_{(1)}|s_{(1)})(u_{(2)}|t_{(1)})(v_{(1)}|s_{(2)})(v_{(2)}|t_{(2)})(w_{(1)}|s_{(3)})(w_{(2)}|t_{(3)})$.

A Laplace pairing is entirely determined by its value on C. In other words, once we know $(a|b)$ for all a and b in C, equation (1) enables us to calculate the Laplace pairing on $S(C)$.

The Laplace pairing induces a twisted product $\circ$ on $S(C)$ by $u\circ v = \sum (u_{(1)}|v_{(1)}) u_{(2)}\cdot v_{(2)}$. By applying the counit to both sides of this equality we obtain the useful relation [3]

$$\varepsilon(u\circ v) = (u|v) \tag{2}$$

If we follow the proofs given in [3] we can easily show that the twisted product is associative, 1 is the unit of $\circ$, $(u\circ v|w) = (u|v\circ w)$ and $\Delta(u\circ v) = \sum u_{(1)}\circ v_{(1)}\otimes u_{(2)}\cdot v_{(2)}$. If we use the last identity recursively, we obtain for $u^1,\ldots,u^k$ in $S(C)$

$$\Delta(u^1\circ\ldots\circ u^k) = \sum u^1_{(1)}\circ\ldots\circ u^k_{(1)}\otimes u^1_{(2)}\cdot\ldots\cdot u^k_{(2)}. \tag{3}$$

This leads us to the important relation

$$u^1\circ\ldots\circ u^k = \sum \varepsilon(u^1_{(1)}\circ\ldots\circ u^k_{(1)}) u^1_{(2)}\cdot\ldots\cdot u^k_{(2)}. \tag{4}$$

To show (4) recursively, we observe that it is true for $k=2$. We denote $U = u^1\circ\ldots\circ u^k$ and we assume that the property is true up to the twisted product of k terms. Since, by definition, $U\circ v = \sum (U_{(1)}|v_{(1)})U_{(2)}\cdot v_{(2)}$, equation (3) yields $U\circ v = \sum (u^1_{(1)}\circ\ldots\circ u^k_{(1)}|v_{(1)})u^1_{(2)}\cdot\ldots\cdot u^k_{(2)}\cdot v_{(2)}$ and the result follows for the twisted product of $k+1$ terms because of equation (2).

Finally, we shall prove the second important identity

$$\varepsilon(u^1\circ\ldots\circ u^k) = \sum \prod_{i=1}^{k-1}\prod_{j=i+1}^{k} (u^i_{(j-1)}|u^j_{(i)}) = \sum\prod_{j>i}(u^i_{(j-1)}|u^j_{(i)}). \tag{5}$$

For $k=2$, equation (5) is true because of equation (2). Assume that it is true up to k and denote $U = u^1\circ\ldots\circ u^k$. From equation (2) and $U = \sum\varepsilon(U_{(1)})U_{(2)}$ we find

$$\begin{aligned}\varepsilon(U\circ u^{k+1}) &= \sum\varepsilon(U_{(1)})(U_{(2)}|u^{k+1}) = \sum\varepsilon(u^1_{(1)}\circ\ldots\circ u^k_{(1)})(u^1_{(2)}\cdot\ldots\cdot u^k_{(2)}|u^{k+1})\\ &= \sum\varepsilon(u^1_{(1)}\circ\ldots\circ u^k_{(1)})\prod_{n=1}^{k}(u^n_{(2)}|u^{k+1}_{(n)}),\end{aligned}$$

where we used equations (3) and (1). Equation (5) is true up to k thus

$$\begin{aligned}\varepsilon(u^1\circ\ldots\circ u^{k+1}) &= \sum\prod_{i=1}^{k-1}\prod_{j=i+1}^{k}(u^i_{(j-1)}|u^j_{(i)})\prod_{n=1}^{k}(u^n_{(k)}|u^{k+1}_{(n)})\\ &= \sum\prod_{i=1}^{k-1}\prod_{j=i+1}^{k+1}(u^i_{(j-1)}|u^j_{(i)})(u^k_{(k)}|u^{k+1}_{(k)}) = \sum\prod_{i=1}^{k}\prod_{j=i+1}^{k+1}(u^i_{(j-1)}|u^j_{(i)})\end{aligned}$$

and the identity is proved for the twisted product of $k+1$ elements.

We considered the symmetric algebra $S(C)$, but the same results are obtained for the tensor algebra $T(C)$. A related construction was made by Hivert in [4].

3. Quantum fields

The previous construction is now applied to interacting quantum field theory. The scalar fields are defined by the usual formula [5]

$$\phi(x) = \int\frac{\mathrm{d}\mathbf{k}}{(2\pi)^3\sqrt{2\omega_k}}\Big(\mathrm{e}^{-ip\cdot x}a(\mathbf{k}) + \mathrm{e}^{ip\cdot x}a^\dagger(\mathbf{k})\Big),$$

where $\omega_k = \sqrt{m^2+|\mathbf{k}|^2}$, $p=(\omega_k,\mathbf{k})$, $a^\dagger(\mathbf{k})$ and $a(\mathbf{k})$ are the creation and annihilation operators acting on the symmetric Fock space of scalar particles. The interaction Lagrangian involves products of fields at the same point. Thus, we define the powers of fields $\phi^n(x)$ as the normal product of n fields at x (i.e. $\phi^n(x) = :\phi(x)\dots\phi(x):$). This definition is meaningful for all $n > 0$ and is extended to $n = 0$ by saying that $\phi^0(x)$ is the unit operator. In the following we shall consider the divided powers of fields defined by $\phi^{(n)}(x) = \phi^n(x)/n!$.

We consider the coalgebra $\mathscr{C}$ generated by $\phi^{(n)}(x)$, where x is a spacetime point and n runs from 0 to 3 for a ϕ^3 theory and from 0 to 4 for a ϕ^4 theory. In this simple case we do not consider derivatives of fields. The coproduct of $\mathscr{C}$ is $\Delta\phi^{(n)}(x) = \sum_{k=0}^{n} \phi^{(k)}(x) \otimes \phi^{(n-k)}(x)$ and its counit is $\varepsilon(\phi^{(n)}(x)) = \delta_{n,0}$. Scalar fields are bosons, so we work with the symmetric algebra $S(\mathscr{C})$. The product of $S(\mathscr{C})$ is the normal product of operators, which is commutative and denoted by $:uv:$. Notice that the counit is equal to the expectation value over the vacuum: $\varepsilon(u) = \langle 0|u|0\rangle$.

In $S(\mathscr{C})$, the Laplace pairing is entirely determined by the value of $(\phi^{(n)}(x)|\phi^{(m)}(y))$, which is itself determined by the value of $(\phi(x)|\phi(y)) = G(x,y)$ if we consider $\phi^{(n)}(x)$ as a product of fields. More precisely $(\phi^{(n)}(x)|\phi^{(m)}(y)) = \delta_{n,m} G(x,y)^{(n)}$, where the right hand side is a divided power $G(x,y)^{(n)} = (1/n!)G(x,y)^n$. In general, $G(x,y)$ is a distribution. In quantum field theory we use two special cases: the Wightman function $G_+(x,y) = \langle 0|\phi(x)\phi(y)|0\rangle$ and the Feynman propagator $G_F(x,y) = \langle 0|T\big(\phi(x)\phi(y)\big)|0\rangle$. As shown in [2], when the Laplace pairing is defined with $G(x,y) = G_+(x,y)$, the twisted product equals the operator product of fields. When it is defined with $G(x,y) = G_F(x,y)$ the twisted product equals the time-ordered product.

Notice that $\Delta^{k-1}\phi^{(n)}(x) = \sum \phi^{(m_1)}(x) \otimes \dots \otimes \phi^{(m_k)}(x)$, with a sum over all nonnegative integers m_i such that $\sum_{i=1}^{k} m_i = n$. Thus, we can specialize equation (1) to our coalgebra $\mathscr{C}$

$$(:\phi^{(n_1)}(x_1)\dots\phi^{(n_k)}(x_k):|:\phi^{(p_1)}(y_1)\dots\phi^{(p_l)}(y_l):) = \sum_M \prod_{i=1}^{k} \prod_{j=1}^{l} G(x_i,y_j)^{(m_{ij})}, \quad (6)$$

where the sum is over all $k \times l$ matrices M of nonnegative integers m_{ij} such that $\sum_{j=1}^{l} m_{ij} = n_i$ and $\sum_{i=1}^{k} m_{ij} = p_j$. This formula was given in reference [6].

Equation (4) applied to $\mathscr{C}$ yields a classical result of quantum field theory

$$\phi^{(n_1)}(x_1) \circ \dots \circ \phi^{(n_k)}(x_k) = \sum_{i_1=0}^{n_1} \cdots \sum_{i_k=0}^{n_k} \langle 0|\phi^{(i_1)}(x_1) \circ \dots \circ \phi^{(i_k)}(x_k)|0\rangle$$
$$:\phi^{(n_1-i_1)}(x_1)\dots\phi^{(n_k-i_k)}(x_k):. \quad (7)$$

This equation was published by Epstein and Glaser for the operator product and the time-ordered product [7]. It is now often used in the Epstein-Glaser approach to renormalisation (see e.g. [8]). Equation (4) is clearly more compact and also more general than equation (7): it is still valid if the elements of $\mathscr{C}$ (i.e. $\phi^{(n_i)}(x_i)$) are replaced by elements of $S(\mathscr{C})$.

Finally, if we specialize equation (5) to $\mathscr{C}$ we obtain

$$\langle 0|\phi^{(n_1)}(x_1) \circ \dots \circ \phi^{(n_k)}(x_k)|0\rangle = \sum_M \prod_{i=1}^{k-1} \prod_{j=i+1}^{k} G(x_i,x_j)^{(m_{ij})}, \quad (8)$$

where the sum is over all symmetric $k \times k$ matrices M of nonnegative integers m_{ij} such that $\sum_{j=1}^{k} m_{ij} = n_j$ and $m_{ii} = 0$ for all i.

When the twisted product is the operator product, this expression was given by Brunetti, Fredenhagen and Köhler [9]. Notice that (8) was proved here with a few lines of algebra, whereas the quantum field proof is combinatorial. As remarked by Rota, a great virtue of Hopf algebras is to replace combinatorics by algebra.

When the twisted product is the time-ordered product, equation (8) has a diagrammatic interpretation. The diagrammatic calculation of $\langle 0|T\big(\phi^{(n_1)}(x_1)\ldots\phi^{(n_k)}(x_k)\big)|0\rangle$ would be: draw all diagrams that have k vertices x_1 to x_k and for which each vertex x_i has n_i edges. There is a one to one correspondence between these diagrams and the matrices M satisfying the conditions stated above: m_{ij} is the number of edges linking vertices x_i and x_j. The condition $m_{ii}=0$ means that there is no tadpoles. The graphs are not directed (i.e. the edges do not carry arrows) because the Feynman propagator $G_F(x,y)$ is symmetric (i.e. $G_F(y,x)=G_F(x,y)$).

4. Perspective

This paper shows that non trivial results of quantum field theory can be derived easily from a general quantum group construction. A word of caution must be added concerning equation (8). When the twisted product is the operator product, equation (8) is valid. It defines a state on $T(\mathscr{C})$ by $\omega(a_1\otimes\ldots\otimes a_k)=\varepsilon(a_1\circ\ldots\circ a_k)$ and the Laplace pairing is a positive semidefinite form on $T(C)\times T(C)$. However, when the twisted product is the time-ordered product, equation (8) is ill-defined because the powers $G_F(x,y)^n$ are singular products of distributions and renormalisation is necessary. The first step of a "quantum group" renormalisation of scalar field theories was done in [2]. It uses the fact that equation (7) is still valid in renormalised quantum field theory, so that the Laplace pairing must be replaced by a Sweedler's 2-cocycle [10] in the definition of a renormalised time-ordered product. To go further, we can implement renormalisation abstractly by starting from a bialgebra B and putting a bialgebra structure on the "squared" tensor algebra $T(T(B)^+)$ [4]. This construction is inspired by Pinter's approach to renormalization [11] and is related to the Faà di Bruno bialgebra of composition of series [10]. These results will be presented in a forthcoming publication.

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