EPJ manuscript No.
(will be inserted by the editor)

Renormalization of QED in an external field

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February 27, 2002

Abstract. The Schwinger equations of QED are rewritten in three different ways as integral equations involving functional derivatives, which are called weak field, strong field, and SCF quantum electrodynamics. The perturbative solutions of these equations are given in terms of appropriate Feynman diagrams. The Green function that is used as an electron propagator in each case is discussed in detail. The general renormalization rules for each of the three equations are provided both in a non perturbative way (Dyson relations) and for Feynman diagrams.

PACS. 12.20.-m Quantum electrodynamics – 11.10.Gh Renormalization

1 Introduction

This paper is a step towards the calculation of photon and electron spectroscopies of matter based on quantum electrodynamics.

Starting from quantum electrodynamics, which is a most accurate and successful theory, garanties that the basis of the calculation is sound. Although such an approach may look too true to be beautiful, it seems adequate because of experimental and theoretical reasons. From the experimental point of view, spectroscopy has made huge advances and is now able to measure tiny effects. From the theoretical point of view, relativistic methods have proved very powerful, even for problems that do no look obviously relativistic. For instance, relativistic density functional theory [1,2] does not meet the nonuniqueness problem of spin-density functional theory [3]. Moreover, relativistic quantum field theory is a safe framework to go beyond the present LDA methods, especially when dealing with excitations.

There are many presentations of QED. We first have to choose the version which is most convenient for molecular or solid-state physics. The main three formalisms used in textbooks are the quantum field, path integrals and Schrödinger representations [4]. The quantum field and Schrödinger approaches manipulate operator-valued distributions [5] and the path integral approach uses ill-defined measures. Therefore, we prefer to use Schwinger's idea Ref.[6] of working directly with Green functions which are given as solutions of equations involving functional derivatives.

This approach has the mathematical advantage of manipulating standard distributions, and the physical advantage of using only measurable quantities, or the more measurable ones.

The standard approach to QED comes clearly from particle physics, where the S matrix is most useful, and where in and out states of the scattering experiments are well defined. In solid state physics, measurements are usually based on a different principle. The spectroscopist shines on the sample a beam of electrons or photons coming from a classical source. By classical we mean that the source is not influenced by the system being measured. After its interaction with the sample, an outgoing beam of electrons or photons is measured. Many spectroscopies can be described within this framework: photoemission, electron scattering, photon scattering, inelastic scattering, BIS, LEED, RHEED, x-ray absorption, UV/visible spectroscopies, etc. We try to stick as much as possible to this experimental point of view.

We describe the photon field with the classical electromagnetic potential (which is not directly measurable but can be calculated in a given gauge from the measurable electric and magnetic fields) and the photon Green function (from which the photon energy density can be computed). We describe the electron field with the one-particle Green function. This is also not directly measurable, but many diagonal matrix elements of the Green function can be measured (e.g. the electronic charge and current densities).

For applications to the spectroscopy of matter, we need to formulate QED with an external field. This external field is made of the potential due to the nuclei, to magnetic or electric fields applied to the sample and to external light sources. After this introduction, we give the notation used for the Green functions and the QED Lagrangian. Then, we derive the Schwinger equations for the electromagnetic potential and the electron propagator. These equations are solved iteratively using three different methods (weak field, strong field, self-consistent field). The boundary conditions, such as the number of electrons in the system, are determined by the unperturbed Green function which is discussed in detail. The proper definition of the current is established through the use of a bilocal operator. Then we describe the renormalization of QED in an external field. The non perturbative Dyson relations are given to express the relation between bare and renormalized Green functions. The renormalization rules for Feynman diagrams are discussed.

The following assumptions are made in the present paper: the external field is weak enough not to create charges (this is true for all stable atomic nuclei [7]) and the external field is zero at infinity (we exclude constant electric and magnetic field, for which specialized monographs are available [8,9]). Moreover, we do not consider IR divergences.

2 Notation

In this section, we specify the notation that is used in the paper. The charge of the electron is $e = -|e|$. The pseudo-metric tensor $g_{\lambda\mu}$ is

$$g = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

We choose the standard gamma matrices ([10] p.693)

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \quad \gamma^j = \begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix}.$$

The Pauli matrices σ^j are defined by

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The charge conjugation and mass reversal matrices are, respectively

$$C = \begin{pmatrix} 0 & -i\sigma^2 \\ -i\sigma^2 & 0 \end{pmatrix}, \quad \text{and} \quad \gamma_5 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}.$$

They satisfy the following identities

$$C = -C^T = -C^\dagger = -C^{-1}, \quad C\gamma_\mu C^{-1} = -\gamma_\mu^T,$$
$$\gamma_5^{-1} = \gamma_5 = \gamma^5, \quad \gamma_5\gamma^\mu\gamma_5 = -\gamma^\mu.$$

2.1 Free propagators

The free photon Green function $D^0_{\mu\nu}(x,y)$ is defined by its Fourier transform

$$D^0_{\mu\nu}(x,y) = \int \frac{\mathrm{d}q}{(2\pi)^4} \exp[-iq.(x-y)] D^0_{\mu\nu}(q),$$

where

$$D^0_{\mu\nu}(q) = -\frac{1}{\epsilon_0(q^2+i\epsilon)}\big(g_{\mu\nu} - (1-1/\xi)q_\mu q_\nu/q^2\big).$$

It is a solution of the equation

$$\epsilon_0\big(\Box g^{\mu\nu} - (1-\xi)\partial^\mu\partial^\nu\big)D^0_{\nu\lambda}(x,y) = \delta^\mu{}_\lambda\delta(x-y).$$

In the last equation the D'Alembertian $\Box$ and the derivatives $\partial^\mu\partial^\nu$ act on variable x.

The free electron propagator is

$$S^0(x,y) = \int \frac{\mathrm{d}q}{(2\pi)^4} \mathrm{e}^{-iq.(x-y)} S^0(q),$$

where

$$S^0(q) = (\hbar c\gamma\cdot q - mc^2 + i\epsilon)^{-1}.$$

It is a solution of the equation

$$(i\hbar c\gamma\cdot\partial - mc^2)S^0(x,y) = \delta(x-y).$$

Analytic expressions for these (and other) propagators are given in [11] and [12] section 2.3. Notice that these Green functions are neither advanced nor retarded. Their physical meaning is discussed in [13,14] and [15] p. 57.

2.2 The QED Lagrangian in SI units

To compare our results with those of nonrelativistic many-body theory, it will be useful to write the QED Lagrangian in SI units [16].

We have $x^0 = ct$, we define the 4-current $J^\alpha = (\rho, \mathbf{J}/c)$, and the 4-potential $A^\alpha = (V, c\mathbf{A})$.

The field-strength tensor is ([10] p.8)

$$F^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha = \begin{pmatrix} 0 & -E^x & -E^y & -E^z \\ E^x & 0 & -cB^z & cB^y \\ E^y & cB^z & 0 & -cB^x \\ E^z & -cB^y & cB^x & 0 \end{pmatrix}.$$

The Maxwell equations are ([10] p.9)

$$\partial_\alpha F^{\alpha\beta} = \frac{1}{\epsilon_0}J^\beta,$$
$$\partial_\alpha \epsilon^{\alpha\beta\gamma\delta}F_{\gamma\delta} = 0.$$

The photon Lagrangian is ([10] p.12)

$$\mathcal{L}_\gamma = -\frac{\epsilon_0}{4}F_{\alpha\beta}F^{\alpha\beta} = \frac{\epsilon_0}{2}(|\mathbf{E}|^2 - c^2|\mathbf{B}|^2).$$

The electron Lagrangian is

$$\mathcal{L}_e = i\hbar c\bar\psi\gamma\cdot\partial\psi - mc^2\bar\psi\psi,$$

where $\gamma\cdot\partial = \gamma^\mu\partial_\mu$. The interaction Lagrangian is

$$\mathcal{L}_I = -J\cdot A = -e\bar\psi\gamma\cdot A\psi.$$

ϵ_0	$L^{-3}M^{-1}T^2C^2$	A^α	$L^{4-d/2}MT^{-2}C^{-1}$
J^α	$L^{1-d}C$	$\mathcal{L}$	$L^{3-d}MT^{-2}$
ψ	$L^{(1-d)/2}$	e	$L^{d/2-2}C$
D	$L^{3-d/2}MT^{-2}C^{-2}$	S	$M^{-1}L^{-d-2}T^2$
Π	$L^{-3-3d/2}M^{-1}T^2C^2$	Σ	$ML^{2-d}T^{-2}$
$\delta\Sigma/\delta a$	$L^{-2-3d/2}C$	ξ	1
$\hbar c$	ML^3T^{-2}	η	$ML^{(5-d)/2}T^{-2}$
I	ML^3T^{-2}		

Table 1. SI units of QED variables.

The current is

$$J_\mu(x) = e\bar\psi(x)\gamma_\mu\psi(x).$$

The total Lagrangian in SI units, including the gauge term is

$$\mathcal{L} = -\frac{\epsilon_0}{4}F_{\alpha\beta}F^{\alpha\beta} + \bar\psi(i\hbar c\gamma\cdot\partial - e\gamma\cdot A - mc^2)\psi \\ -\frac{\epsilon_0\xi}{2}(\partial\cdot A)^2.$$

2.3 Dimensions

Table 1 gives the SI units of the quantities used in the paper, in a spacetime with dimensions d ($d=4$ for standard spacetime).

3 Derivation of the functional equations

The Schwinger equations were presented in Ref.[6], and various derivations of them are available [17],[18] (p. 416-32) and [10] (p. 475-81). Our derivation follows Ref.[10].

We define a generating function $Z = Z(j,\eta,\bar\eta)$, where the photon source $j_\mu(x)$ and the anticommuting electron sources $\eta(x)$ and $\bar\eta(x)$ enter the total action as [10]

$$\sigma = I - \int \mathrm{d}x\, j_\mu(x)A^\mu(x) + \int \mathrm{d}x\,\bar\eta(x)\psi(x) + \bar\psi(x)\eta(x),$$

where $I = \int \mathrm{d}x\mathcal{L}(x)$. The minus sign before $j_\mu(x)$ in the definition of σ was chosen so that $j_\mu(x)$ is a standard electromagnetic current. It is the opposite of the convention used in Ref.[10].

The generating function can be written, up to a normalization factor, as a path integral ([10] p.476)

$$Z(j,\bar\eta,\eta) = \int \mathcal{D}(A,\psi,\bar\psi)\mathrm{e}^{i\sigma/\hbar c}, \tag{1}$$

or as the mean value of an operator ([10] p.210, 261)

$$Z(j,\bar\eta,\eta) = \langle T\exp\big(i(\sigma - I)/\hbar c\big)\rangle.$$

It is also possible to write $Z(j,\bar\eta,\eta)$ in terms of a $Z_0(j,\bar\eta,\eta)$ without interaction [10] p. 445 and [19] p.246. Anyway, we do not calculate Z, we only need it to calculate mean values of Heisenberg operators. For instance, the mean value $\langle A_\mu(x)\rangle$ of the Heisenberg operator for the electromagnetic potential is given by

$$Z\,\langle A_\mu(x)\rangle = \int \mathcal{D}(A,\psi,\bar\psi)A_\mu(x)\mathrm{e}^{i\sigma/\hbar c}. \tag{2}$$

From the definition of σ, $A_\mu(x)$ can be written as a functional derivative with respect to $j^\mu(x)$:

$$A_\mu(x) = -\frac{\delta\sigma}{\delta j^\mu(x)}.$$

Thus Eq.(2) becomes

$$Z\langle A_\mu(x)\rangle = -\int \mathcal{D}(A,\psi,\bar\psi)\frac{\delta\sigma}{\delta j^\mu(x)}\mathrm{e}^{i\sigma/\hbar c}. \tag{3}$$

Comparing the right hand side of Eq.(3) with $\delta Z/\delta j^\mu(x)$ we obtain

$$Z\langle A_\mu(x)\rangle = i\hbar c\frac{\delta Z}{\delta j^\mu(x)}. \tag{4}$$

Similarly, we shall use

$$Z\langle T\psi_\alpha(x)\bar\psi_\beta(y)\rangle = (\hbar c)^2\frac{\delta^2 Z}{\delta\eta^\beta(y)\delta\bar\eta^\alpha(x)},$$

and

$$Z\langle TA_\lambda(x)A_\mu(y)\rangle = -(\hbar c)^2\frac{\delta^2 Z}{\delta j^\mu(y)\delta j^\lambda(x)}.$$

Finally, the following property will be essential

$$\langle\psi(x)\rangle = \langle\bar\psi(x)\rangle = 0.$$

These equations are derived from the fact that the ground state of the system is an eigenstate of the charge. If Q is the charge operator, then $[Q,\psi(x)] = -\psi(x)$ ([10] p.147), thus

$$\langle\psi(x)\rangle = \langle G|\psi(x)|G\rangle = -\langle G|Q\psi(x)|G\rangle + \langle G|\psi(x)Q|G\rangle.$$

The ground state is an eigenstate of Q with eigenvalue N, thus

$$\langle G|\psi(x)|G\rangle = -N\langle G|\psi(x)|G\rangle + N\langle G|\psi(x)|G\rangle = 0.$$

More physically, no anticommuting operator is measurable, only products of an even number of anticommuting operators can be observed. It may be stressed that the mean values of $\psi(x)$ and $\bar\psi(x)$ are zero only when the (unphysical) electron sources $\eta(x)$ and $\bar\eta(x)$ are set to zero.

3.1 The photon equation

To derive the Schwinger equations we use the fact that the derivative of an integral is zero (assuming that the integrand is zero at infinity):

$$\int \mathcal{D}(A,\psi,\bar\psi)\frac{\delta\mathrm{e}^{i\sigma/\hbar c}}{\delta A^\mu(x)} = 0.$$

Thus

$$\int \mathcal{D}(A,\psi,\bar\psi)\Big(\frac{\delta I}{\delta A^\mu(x)} - j_\mu(x)\Big)\mathrm{e}^{i\sigma/\hbar c} = 0. \qquad (5)$$

A direct calculation leads to

$$\frac{\delta I}{\delta A^\mu(x)} = \epsilon_0\big(\Box g_{\mu\nu} - (1-\xi)\partial_\mu\partial_\nu\big)A^\nu(x) \\ -e\bar\psi(x)\gamma_\mu\psi(x). \qquad (6)$$

According to the method of generating functions, a factor $A^\nu(x)$ in the integral can be replaced by a functional derivative of Z with respect to $j_\nu(x)$ (up to a factor $i\hbar c$). This is what we did for Eq.(4). In the case of anticommuting variables such as $\psi(x)$ we must be a little more careful. Functional derivative with respect to an anticommuting source is very similar to that with respect to a function. The difference can be summarized in the identity

$$\frac{\delta}{\delta\eta}(AB) = \frac{\delta A}{\delta\eta}B + (-1)^{|A|}A\frac{\delta B}{\delta\eta},$$

where A is a product of $|A|$ anticommuting variables. For instance, $|A_\mu| = 0$, $|\psi| = 1$, $|\bar\eta| = 1$, $|\bar\psi\gamma^\mu\psi| = 2$, etc.

Thus, each factor $\psi(x)$ (resp. $\bar\psi(x)$) in the path integral is replaced by a functional derivative of Z with respect to $\bar\eta(x)$ (resp. $\eta(x)$). Paying attention to the signs and the factors $\hbar c/i$ we can rewrite Eq.(5) as

$$\frac{\delta I}{\delta A^\mu(x)}\Big(\frac{-\hbar c\delta}{i\delta j}, \frac{\hbar c\delta}{i\delta\bar\eta}, \frac{-\hbar c\delta}{i\delta\eta}\Big)Z = Zj_\mu(x). \qquad (7)$$

Therefore, Eqs.(7) and (6) yield

$$\epsilon_0\big(\Box g_{\mu\nu} - (1-\xi)\partial_\mu\partial_\nu\big)\frac{-\hbar c\delta Z}{i\delta j_\nu(x)} \\ -e\frac{-\hbar c\delta}{i\delta\eta^s(x)}\gamma_\mu^{ss'}\frac{\hbar c\delta Z}{i\delta\bar\eta^{s'}(x)} = Zj_\mu(x).$$

The value of the vector potential and the electron wavefunctions are obtained by [10]

$$A_\mu(x) = -\frac{\hbar c}{i}\frac{1}{Z}\frac{\delta Z}{\delta j^\mu(x)}, \qquad (8)$$

$$\psi_s(x) = \frac{\hbar c}{i}\frac{1}{Z}\frac{\delta Z}{\delta\bar\eta^s(x)}, \qquad (9)$$

$$\bar\psi_s(x) = -\frac{\hbar c}{i}\frac{1}{Z}\frac{\delta Z}{\delta\eta^s(x)}. \qquad (10)$$

To simplify the notation, we have written $A_\mu(x)$, $\psi_s(x)$ and $\bar\psi_s(x)$ for $\langle A_\mu(x)\rangle$, $\langle\psi_s(x)\rangle$ and $\langle\bar\psi_s(x)\rangle$.

Thus, we obtain

$$Z\epsilon_0\big(\Box g_{\mu\nu} - (1-\xi)\partial_\mu\partial_\nu\big)A^\nu(x) \\ -ei\hbar c\frac{\delta}{\delta\eta^s(x)}\gamma_\mu^{ss'}Z\psi_{s'}(x) = Zj_\mu(x).$$

We define now the electron Green function by

$$S_{ss'}(x,y) = i\hbar c\frac{\delta^2\log Z}{\delta\eta^{s'}(y)\delta\bar\eta^s(x)} \\ = -\frac{\delta\psi_s(x)}{\delta\eta^{s'}(y)} = -\frac{\delta\bar\psi_{s'}(y)}{\delta\bar\eta^s(x)}. \qquad (11)$$

Thus, the equation becomes

$$\epsilon_0\big(\Box g_{\mu\nu} - (1-\xi)\partial_\mu\partial_\nu\big)A^\nu(x) \\ -e\bar\psi(x)\gamma_\mu\psi(x) + ie\hbar c\,\mathrm{tr}[\gamma_\mu S(x,x)] = j_\mu(x).$$

Notice that $S(x,y)$ is singular at $x = y$ and $S(x,x)$ must be regularized (see section 6).

When the external electron sources η and $\bar\eta$ are set to zero, we showed that $\psi(x) = \bar\psi(x) = 0$, and we obtain our first basic equation

$$\epsilon_0\big(\Box g_{\mu\nu} - (1-\xi)\partial_\mu\partial_\nu\big)A^\nu(x) \\ +ie\hbar c\,\mathrm{tr}[\gamma_\mu S(x,x)] = j_\mu(x). \qquad (12)$$

It is a bit clumsy to use an electron source $\eta(x)$ just to conclude that no such source exists which leads to the cancelation of $\psi(x)$ and $\bar\psi(x)$. The way out of this difficulty is to use Rochev's bilocal source $\eta(x,y)$ [20].

$$\sigma = I - \int \mathrm{d}x j_\mu(x)A^\mu(x) + \int \mathrm{d}x\mathrm{d}y\bar\psi(x)\eta(x,y)\psi(y),$$

where $\eta(x,y)$ is now a physically reasonable source of electron-positron pairs. Such a source would lead immediately to Eq.(12). We used the more standard electron sources to follow the textbook derivations [10].

Equation (12) means physically that an induced current $-ie\hbar c\,\mathrm{tr}[\gamma_\mu S(x,x)]$ must be added to the external current $j_\mu(x)$ as a source of electromagnetic potential.

3.2 The electron equation

The second equation is obtained by varying $\bar\psi(x)$. Following [10] p.478 we obtain

$$\frac{\delta I}{\delta\bar\psi(x)}\Big(\frac{-\hbar c\delta}{i\delta j}, \frac{\hbar c\delta}{i\delta\bar\eta}, \frac{-\hbar c\delta}{i\delta\eta}\Big)Z = -Z\eta(x). \qquad (13)$$

The functional derivative of the action yields

$$\frac{\delta I}{\delta\bar\psi(x)} = (i\hbar c\gamma\cdot\partial - mc^2)\psi(x) - e\gamma\cdot A(x)\psi(x). \quad (14)$$

Introducing this into Eq.(13), we obtain

$$-Z\eta(x) = (i\hbar c\gamma\cdot\partial - mc^2)\frac{\hbar c\delta Z}{i\delta\bar\eta(x)} - e\gamma_\mu\frac{-\hbar c\delta}{i\delta j_\mu(x)}\frac{\hbar c\delta Z}{i\delta\bar\eta(x)}.$$

From Eq.(9) we can write

$$\frac{\hbar c}{i}\frac{\delta Z}{\delta\bar\eta^s(x)} = Z\psi_s(x).$$

Hence

$$-\eta(x) = (i\hbar c\gamma\cdot\partial - mc^2)\psi(x) - ie\hbar c\gamma_\mu \frac{\delta\psi(x)}{\delta j_\mu(x)} \\ -e\gamma\cdot A(x)\psi(x).$$

Finally, we take the functional derivative of this equation with respect to $\eta(y)$, we use Eq.(11), the fact that $\psi(x)=0$ and we obtain our second basic equation

$$\delta(x-y) = (i\hbar c\gamma\cdot\partial - mc^2 - e\gamma\cdot A(x))S(x,y) \\ -ie\hbar c\gamma_\mu \frac{\delta S(x,y)}{\delta j_\mu(x)}. \qquad (15)$$

Again, the same result can be obtained without electron sources η, $\bar\eta$ by using the electron-positron source $\eta(x,y)$ [20].

The equation for the electromagnetic vector potential is

$$A_\mu(x) = a_\mu(x) - ie\hbar c\int \mathrm{d}s D^0_{\mu\nu}(x,s)\mathrm{tr}[\gamma^\nu S(s,s)], \quad (16)$$

where $a_\mu(x)$ is the external potential created by the external current $j_\nu(x)$:

$$a_\mu(x) = \int \mathrm{d}y D^0_{\mu\nu}(x,y) j^\nu(y).$$

In terms of the external potential $a_\mu(x)$, the functional equation (15) becomes

$$(i\hbar c\gamma\cdot\partial - mc^2 - e\gamma\cdot A(x))S(x,y) = \delta(x-y) \\ +ie\hbar c\gamma^\mu\int \mathrm{d}s \frac{\delta S(x,y)}{\delta a_\lambda(s)} D^0_{\lambda\mu}(s,x). \qquad (17)$$

On the other hand, we can take the functional derivative of the action with respect to $\psi(x)$

$$\frac{\delta I}{\delta\psi(x)} = i\hbar c\partial^\mu\bar\psi(x)\gamma_\mu + \bar\psi(x)(e\gamma\cdot A^\mu(x) + mc^2). (18)$$

We can now repeat the calculation that was done starting from Eq.(14). This gives us another equation for the electron Green function

$$\delta(x-y) = -i\hbar c\partial^\mu_x S(y,x)\gamma_\mu - S(y,x)(e\gamma\cdot A(x) + mc^2) \\ -ie\hbar c\frac{\delta S(y,x)}{\delta j_\mu(x)}\gamma_\mu. \qquad (19)$$

3.3 Photon Green function

For spectroscopic applications, it is useful to know the photon Green function

$$D_{\mu\nu}(x,y) = \frac{\delta A_\mu(x)}{\delta j^\nu(y)}.$$

In physical terms, the photon Green function gives the linear response of the electromagnetic potential to the variation $\delta j^\nu(y)$ of the external source [1] by

$$\delta A_\mu(x) = \int \mathrm{d}y D_{\mu\nu}(x,y)\delta j^\nu(y).$$

To obtain an equation for $D_{\mu\nu}(x,y)$, we solve Eq.(12) for $A_\mu(x)$.

$$A_\mu(x) = \int \mathrm{d}z D^0_{\mu\lambda}(x,z)\Big(j^\lambda(z) - ie\hbar c\mathrm{tr}[\gamma^\lambda S(z,z)]\Big).$$

A functional derivative with respect to $j^\nu(y)$ gives us the equation for $D_{\mu\nu}(x,y)$

$$D_{\mu\nu}(x,y) = D^0_{\mu\nu}(x,y) - ie\hbar c\int \mathrm{d}z D^0_{\mu\lambda}(x,z)\mathrm{tr}[\gamma^\lambda\frac{\delta S(z,z)}{\delta j^\nu(y)}]. \qquad (20)$$

4 Three integral equations

In this section, we derive various integral equations which correspond to the differential equations of the previous section.

4.1 Weak external potential

When the external potential is weak, we can multiply Eq.(17) by the free electron Green function $S^0(x,y)$. This gives us the following coupled equations

$$S(x,y) = S^0(x,y) + e\int \mathrm{d}z S^0(x,z)\gamma\cdot A(z)S(z,y) \\ +ie\hbar c\int \mathrm{d}z\mathrm{d}z' S^0(x,z)\gamma^\mu\frac{\delta S(z,y)}{\delta a_\nu(z')}D^0_{\nu\mu}(z',z), \\ A_\mu(x) = a_\mu(x) - ie\hbar c\int \mathrm{d}y D^0_{\mu\nu}(x,y)\mathrm{tr}[\gamma^\nu S(y,y)].$$

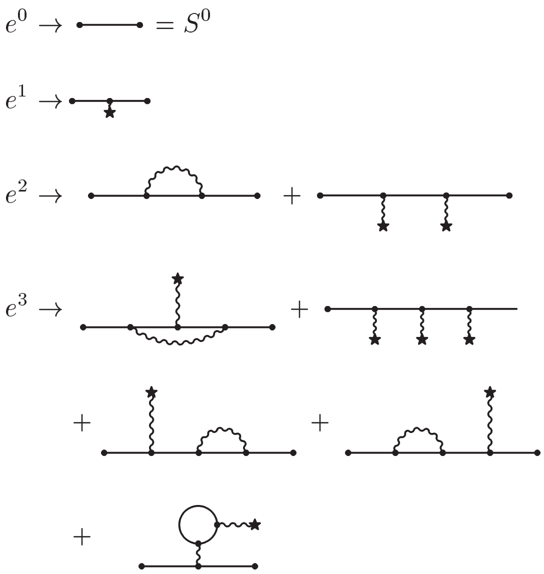
The weak field approach to atomic physics was reviewed recently by Eides and coll. [22].

4.1.1 Feynman diagrams

In the Feynman diagram representation, we denote the external current $j_\mu(x)$ by a star $\star$, so the external potential $a_\mu(x)$ is denoted by the diagram .

The coupled equations generate the following series for the electron Green function. The propagator is oriented from right to left and the electron loops are oriented anticlockwise.

[1] In fact, the physical vector potential is given by the retarded photon Green function, which is different from D, but this is a standard problem [21].



4.2 Strong external potential

When the external potential is strong, we multiply Eq.(17) by the Green function in the presence of $a_\mu(x)$:

$$S^a = \big(i\hbar c\gamma \cdot \partial - mc^2 - e\gamma \cdot a(x)\big)^{-1}.$$

Now the coupled equations become

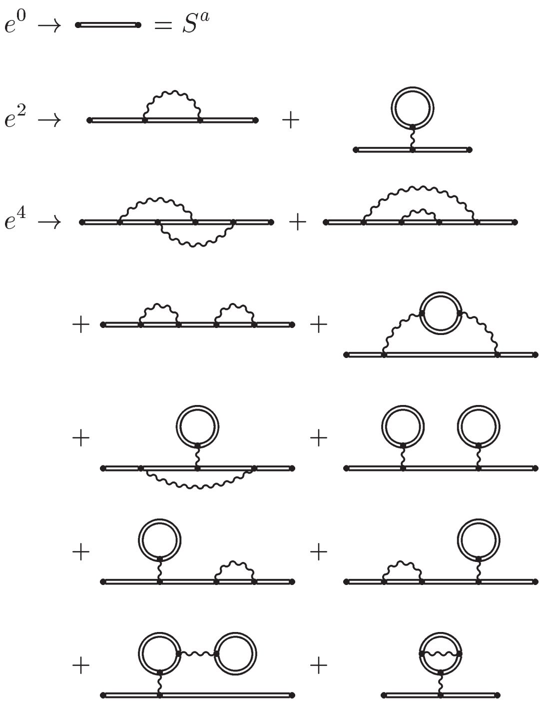
$$\begin{aligned} S(x,y) &= S^a(x,y) + e\int \mathrm{d}z S^a(x,z)\gamma \cdot A'(z)S(z,y) \\ &\quad + ie\hbar c\int \mathrm{d}z\mathrm{d}z' S^a(x,z)\gamma^\mu \frac{\delta S(z,y)}{\delta a_\nu(z')} D^0_{\nu\mu}(z',z), \\ A'_\mu(x) &= -ie\hbar c\int \mathrm{d}s D^0_{\mu\nu}(x,y)\mathrm{tr}[\gamma^\nu S(y,y)]. \end{aligned}$$

In this case, the total potential is $A_\mu(x) = a_\mu(x) + A'_\mu(x)$.

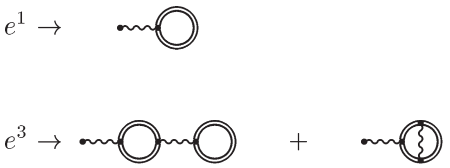
Strong field QED was reviewed recently in Ref. [23]. Strong field QED can easily accomodate bound states, but the nuclear potential is not screened.

4.2.1 Feynman diagrams

For the electron Green functions, the propagator is oriented from right to left and the electron loops are oriented anticlockwise, except for the tadpoles, where a loop diagram is half the sum of a clockwise loop and an anticlockwise loop.



For the potential $A^\mu(x)$ we have the following Feynman diagrams



If s_n (resp. a_n) denotes the number of Feynman diagrams for $S(x,y)$ (resp $A^\mu(x)$) at order e^n, the coupled equations yield

$$\begin{aligned} s_0 &= 1, \quad a_0 = 0, \\ a_n &= s_{n-1}, \\ s_n &= \sum_{i=0}^{n-1} s_i a_{n-i-1} + (n-1)s_{n-2}. \end{aligned}$$

Thus, $s_{2n} = 1, 2, 10, 74, 706, 8162, 110410, ...$ and $s_{2n+1} = 0$. If $s(x)$ is the generating function for the sequence s_n, then

$$s(x) = \sum_{n=0}^{\infty} s_n x^n = \frac{f(x)}{1 + x^2 f(x)},$$

where

$$f(x) = \sum_{n=0}^{\infty} (2n+1)!!\, x^{2n}.$$

For similar results, see [10] p.467.

4.3 SCF external potential

Finally, the most accurate results are obtained when the external field is taken as the complete vector potential $A_\mu(x)$, which is determined by a self-consistent field procedure. Therefore, the potential that will be used in the initial Green function is $A_\mu(x)$ instead of $a_\mu(x)$. This calculation can be found in Refs.[18] and [10] p.479.

To eliminate the external current $j^\nu(y)$ from Eq.(15) we need the functional relation

$$\begin{aligned}\frac{\delta S(x,z)}{\delta j^\nu(y)} &= \int \mathrm{d}s \frac{\delta S(x,z)}{\delta A_\mu(s)}\frac{\delta A_\mu(s)}{\delta j^\nu(y)} \\ &= \int \mathrm{d}s \frac{\delta S(x,z)}{\delta A_\mu(s)} D_{\mu\nu}(s,y).\end{aligned}$$

With this relation, the functional equation for the photon Green function Eq.(20) becomes

$$\begin{aligned}D_{\mu\nu}(x,y) &= D^0_{\mu\nu}(x,y) - ie\hbar c\int \mathrm{d}z\mathrm{d}s D^0_{\mu\lambda}(x,z) \\ &\quad\times \mathrm{tr}[\gamma^\lambda \frac{\delta S(z,z)}{\delta A_{\lambda'}(s)}] D_{\lambda'\nu}(s,y), \qquad (21)\end{aligned}$$

and the functional equation for the electron Green function Eq.(15) becomes

$$\begin{aligned}(i\hbar c\gamma\cdot\partial - mc^2 - e\gamma\cdot A(x))S(x,y) &= \delta(x-y) \\ +ie\hbar c\gamma^\mu \int \mathrm{d}s \frac{\delta S(x,y)}{\delta A_\lambda(s)} D_{\lambda\mu}(s,x). &\qquad (22)\end{aligned}$$

Then, we multiply Eq.(22) by the Green function in the presence of $A_\mu(x)$:

$$S^A = \big(i\hbar c\gamma\cdot\partial - mc^2 - e\gamma\cdot A(x)\big)^{-1},$$

and the equation for the electron Green function becomes

$$\begin{aligned}S(x,y) &= S^A(x,y) \\ &+ie\hbar c \int \mathrm{d}z\mathrm{d}z' S^A(x,z)\gamma_\mu \frac{\delta S(z,y)}{\delta A^\nu(z')} D^{\nu\mu}(z',z). \, (23)\end{aligned}$$

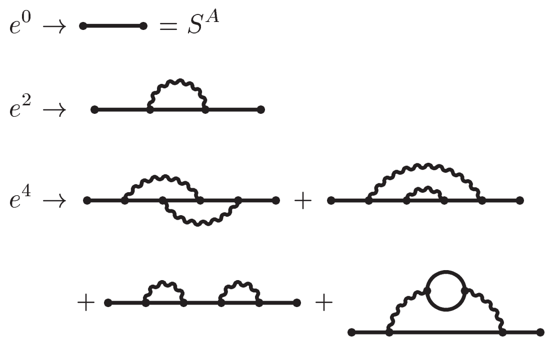
This equation, together with Eq.(21) and Eq.(16) are a complete system of equations for the determination of the bare Green functions and the potential.

As compared to the case of a strong external potential, the SCF potential has the advantage of taking into account the electrons in the system. Thus, the nuclear potentials are screened by the electrons in a self-consistent way. All the tadpoles of strong field QED have been resummed.

This formulation of QED is closer to the standard methods of solid state physics or quantum chemistry. It was used in nuclear physics under the name Hartree QED when the current is calculated by a single electron loop [24], see also [25].

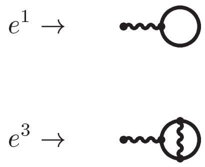
4.3.1 Electron Green function for self-consistent field

For the electron Green functions, all electron loops are oriented anticlockwise and the propagator is oriented from right to left.



The number of Feynman diagrams at each order was calculated in Ref.[26] (see also [27]).

For the potential, we have the following diagrams



Each strong field diagram is the sum of an infinite number of weak field diagrams, and each SCF diagram is the sum of an infinite number of strong field diagrams.

5 The initial Green function

Once the functional equation is know, it remains to specify the boundary conditions. For example, we can specify the Green function without interaction. In the case of a static external field $a_\mu(\mathbf{r})$ or a static self-consistent field $A_\mu(\mathbf{r})$, this Green function is well-known and can be written

$$\begin{aligned}G^0_N(x,y) &= i\theta(y^0-x^0)\sum_{E_n\le E_F}\psi_n(x)\bar\psi_n(y) \\ &-i\theta(x^0-y^0)\sum_{E_n> E_F}\psi_n(x)\bar\psi_n(y), \quad (24)\end{aligned}$$

where E_F is the Fermi energy, and where $\psi_n(x)$ is the solution of the Dirac equation at energy E_n. Note that $G^0_N(x,y) = \hbar c S^a_N(x,y)$ or $G^0_N(x,y) = \hbar c S^A_N(x,y)$ where $S^a_N(x,y)$ and $S^A_N(x,y)$ are the Feynman electron Green functions in the external potential $a_\mu(x)$ or $A_\mu(x)$ (with N occupied bound states). In the present section, we want to discuss and justify this expression.

The notation follows standard textbooks on the Dirac equation [10,28]. The index N means that $G^0_N(x,y)$ is the Green function for a total charge of Ne. The Green function $G_0^0(x,y)$ represents the vacuum in the presence of $a_\mu(\mathbf{x})$ (or $A_\mu(\mathbf{x})$). This vacuum Green function is given by Eq.(24) for $E_F = 0$. The difference between $G_N^0(x,y)$ and $G_0^0(x,y)$ is

$$\begin{aligned} G_N^0(x,y) - G_0^0(x,y) &= i\theta(y^0 - x^0) \sum_{0<E_n\le E_F} \psi_n(x)\bar{\psi}_n(y) \\ &+ i\theta(x^0 - y^0) \sum_{0<E_n\le E_F} \psi_n(x)\bar{\psi}_n(y), \\ &= i \sum_{0<E_n\le E_F} \psi_n(x)\bar{\psi}_n(y). \end{aligned}$$

The Fermi energy E_F is determined by the condition that the sum is over N states:

$$\sum_{0<E_n\le E_F} 1 = N.$$

5.1 Early works

To understand the origin of the Green function given by Eq.(24), it is useful to describe how it appeared historically.

In the vacuum, Dirac [29,30] assumed that all the negative-energy states are filled. This gives the density matrix

$$\rho(\mathbf{x},\mathbf{y}) = \sum_{E_n<0} \psi_n^0(\mathbf{x}){\psi_n^0}^\dagger(\mathbf{y}),$$

where $\psi_n^0(\mathbf{x})$ is the free solution of the Dirac equation for energy E_n. However, such a density matrix would be highly impractical, since it would lead to an infinite charge density. In a strike of genius, Dirac made the following observation [29],[30]. Since

$$\sum_{E_n} \psi_n^0(\mathbf{x}){\psi_n^0}^\dagger(\mathbf{y}) = \delta(\mathbf{x}-\mathbf{y}),$$

the "renormalized" density matrix

$$\begin{aligned} \rho_0(\mathbf{x},\mathbf{y}) &= \rho(\mathbf{x},\mathbf{y}) - \frac{1}{2}\delta(\mathbf{x}-\mathbf{y}) \\ &= \frac{1}{2}\sum_{E_n<0} \psi_n^0(\mathbf{x}){\psi_n^0}^\dagger(\mathbf{y}) - \frac{1}{2}\sum_{E_n>0} \psi_n^0(\mathbf{x}){\psi_n^0}^\dagger(\mathbf{y}) \end{aligned}$$

is much better behaved.[2] In particular, the charge density is

$$\begin{aligned} \rho_0(\mathbf{x}) &= \mathrm{tr}[\rho_0(\mathbf{x},\mathbf{x})] \\ &= \frac{1}{2}\sum_{E_n<0} |\psi_n^0(\mathbf{x})|^2 - \frac{1}{2}\sum_{E_n>0} |\psi_n^0(\mathbf{x})|^2 = 0, \end{aligned}$$

where the last equality is due to charge conjugation symmetry.

[2] In fact, Dirac defined $\rho_0(\mathbf{x},\mathbf{y})$ as the double of the right-hand side. This was immediately corrected by Heisenberg [31].

In the presence of a static potential $a_\mu(\mathbf{x})$ or $A_\mu(\mathbf{x})$, and without bound states, the analogous density matrix is

$$\rho(\mathbf{x},\mathbf{y}) = \frac{1}{2}\sum_{E_n<0} \psi_n(\mathbf{x}){\psi_n}^\dagger(\mathbf{y}) - \frac{1}{2}\sum_{E_n>0} \psi_n(\mathbf{x}){\psi_n}^\dagger(\mathbf{y}),$$

where $\psi_n(\mathbf{x})$ is now the solution of the Dirac equation with a potential $a_\mu(\mathbf{x})$ or $A_\mu(\mathbf{x})$ and energy E_n. Of course, the charge conjugation symmetry is broken by the external potential, but we still have that

$$\begin{aligned} \rho(\mathbf{x}) &= \mathrm{tr}\big(\rho(\mathbf{x},\mathbf{x})\big) = \mathrm{tr}\big(\rho(\mathbf{x},\mathbf{x})\big) - \rho_0(\mathbf{x}) \\ &= \frac{1}{2}\sum_{E_n<0} (|\psi_n(\mathbf{x})|^2 - |\psi_n^0(\mathbf{x})|^2) \\ &- \frac{1}{2}\sum_{E_n>0} (|\psi_n(\mathbf{x})|^2 - |\psi_n^0(\mathbf{x})|^2) \end{aligned}$$

is less singular. The physical idea developed by Dirac is to measure the charge density with respect to the charge density of the vacuum $\rho_0(\mathbf{x})$. The induced charge density $\rho(\mathbf{x}) \neq 0$ in the presence of an external potential was investigated in the thirties [29], [30], [31], [32], [33]. The potential corresponding to this induced charge is called the Uehling potential [34]. Physically, $\rho(\mathbf{x}) \neq 0$ is a reaction of the vacuum which is polarized by the external potential. The induced charge has observable consequences. For instance, its effect on the energy levels of hydrogen atoms is well documented [23]. Computer programs are available to evaluate the Uehling potential for general nuclear charge models [35]. Expansions in $(Z\alpha)^n$ where investigated in [36].

5.2 Checking the Green function

In this section, we check that Eq.(24) gives the charge density proposed by Dirac. In Eq.(24) we write

$$\psi_n(x) = \exp(-iE_n x^0/c\hbar)\psi_n(\mathbf{x}),$$

and we define the frequency dependent Green function

$$G_N^0(\mathbf{x},\mathbf{y};\omega) = \int_{-\infty}^{\infty} \mathrm{d}t\, \mathrm{e}^{i\omega t} G_N^0(x,y),$$

where, on the right hand side, $x^0 = y^0 + ct$. Since the external potential is time independent, the Green function depends only on t. To determine $G_N^0(\mathbf{x},\mathbf{y};\omega)$ we use the distribution identity (Ref.[10], p.92)

$$\theta(x^0 - y^0) = i\int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi} \frac{\exp[-i\omega(x^0-y^0)]}{\omega + i\epsilon},$$

and we obtain

$$\begin{aligned} G_N^0(\mathbf{x},\mathbf{y};\omega) &= \sum_{E_n\le E_F} \frac{\psi_n(\mathbf{x})\bar{\psi}_n(\mathbf{y})}{\omega - E_n/\hbar - i\epsilon} \\ &+ \sum_{E_n>E_F} \frac{\psi_n(\mathbf{x})\bar{\psi}_n(\mathbf{y})}{\omega - E_n/\hbar + i\epsilon}. \end{aligned}$$

This is the standard expression for the non-interacting electron Green function in the relativistic case (see Ref.[23] p. 32 for $E_F = 0$) and as well as in the nonrelativistic case (up to a factor γ^0, see Ref. [37], p.124).

The induced current is $-ie\mathrm{tr}[\gamma^0 G_N^0(x,x)]$. But, this formal expression is infinite and must be regularized. The limit involved in the expression $G_N^0(x,x)$ is discussed in detail in section 6:

$$G_N^0(x,x) = \frac{1}{2}\lim_{\mathbf{z}\to 0}\lim_{z^0\to 0^+}\big(G_N^0(x+z,x-z) + G_N^0(x-z,x+z)\big).$$

This definition agrees with experiments in nuclear physics [7] and atomic physics (Ref.[23] p. 275).

With this definition, we obtain

$$G_N^0(x,x) = \frac{i}{2}\Big(\sum_{E_n\le E_F}\psi_n(\mathbf{x})\bar\psi_n(\mathbf{x}) - \sum_{E_n> E_F}\psi_n(\mathbf{x})\bar\psi_n(\mathbf{x})\Big).$$

In particular, we obtain the charge density:

$$\rho(\mathbf{x}) = J^0(\mathbf{x}) = -ie\mathrm{tr}[\gamma^0 G_N^0(x,x)] = \frac{e}{2}\Big(\sum_{E_n\le E_F}|\psi_n(\mathbf{x})|^2 - \sum_{E_n> E_F}|\psi_n(\mathbf{x})|^2\Big).$$

This is indeed the density obtained by Dirac. One might wonder how this charge density is related to the many-body charge density (or the non-relativistic one), which is obtained by summing over the occupied states with positive energy. The answer is that we must subtract the contribution of the Green function $G_0^0(x,x)$ (for which $E_F = 0$). Then we obtain

$$-ie\mathrm{tr}[\gamma^0\big(G_N^0(x,x) - G_0^0(x,x)\big)] = e\sum_{0<E_n\le E_F}|\psi_n(\mathbf{x})|^2.$$

The right hand side is the familiar expression for $\rho(\mathbf{x})$. It is not strictly correct, since it neglects the effect of $G_0^0(x,x)$, which describes the polarisation of the vacuum by the external potential (i.e. the Uehling potential, see Ref.[10] p. 327 or Ref.[23] p.265). It might seem strange that the QED and the many-body densities are different, since they are given by the same equation

$$\rho(\mathbf{x}) = -\frac{ie}{2\pi}\int_{-\infty}^{\infty}\mathrm{d}\omega\,\mathrm{tr}[\gamma^0 G_N^0(\mathbf{x},\mathbf{x};\omega)].$$

The reason for this discrepancy is that the expression for $\rho(\mathbf{x})$ is divergent, and the difference comes from a different renormalization of it. In the many-body case, an innocent-looking convergence factor $\mathrm{e}^{i\omega\eta}$ is introduced in the integral over ω (see Ref.[37] p.124), which selects precisely the states $E_n \le E_F$ (the states $E_n < 0$ are discarded from the start). On the other hand, in the QED case, the integration contour is below the real axis from $-\infty$ to E_F and above it from E_F to ∞.

Several approximations are possible (no-sea, no-pair). They are discussed in Refs.[38,39].

5.3 Subsequent works

This idea of a Dirac sea in an external potential has received many experimental and theoretical confirmations. The first and strongest one being the discovery of the positron.

Since then, the Dirac sea and Eq.(24) were used as a basis for strong field electrodynamics [7,40–42], for thermal field theory [43–46], and for relativistic quantum many-body theory in Riemannian spacetime [47].

The concept of a Dirac sea is sometimes considered to be rather out of date. However, according to Jackiw [48], it is still the best physical picture to understand striking phenomena such as chiral anomalies [49] and fractional fermion numbers [50].

The case when the Fermi energy is degenerate will not be considered in this paper.

6 The induced current

It is important to derive the proper expression of the four-current as a function of the electron Green function. As a field operator, the current must be self-adjoint (it is measurable), conserved, gauge invariant and should change sign under charge conjugation.

6.1 Definition

We follow Refs.[15,51,52] and define the bilocal field operator

$$\mathbf{J}^\mu(x,y) = \frac{e}{2}\sum_{\alpha\beta}\gamma^\mu_{\alpha\beta}\big[\bar{\boldsymbol\psi}_\alpha(x)\boldsymbol\psi_\beta(y) - \boldsymbol\psi_\beta(x)\bar{\boldsymbol\psi}_\alpha(y)\big].$$

When this operator is evaluated in the ground state of the system, its value can be expressed in terms of the electron Green function.

The Stueckelberg-Feynman Green functions that we use in the present paper are defined from the time-ordering operator ([10] p.150)

$$T\boldsymbol\psi_\alpha(x)\bar{\boldsymbol\psi}_\beta(y) = \theta(x^0-y^0)\boldsymbol\psi_\alpha(x)\bar{\boldsymbol\psi}_\beta(y) - \theta(y^0-x^0)\bar{\boldsymbol\psi}_\beta(y)\boldsymbol\psi_\alpha(x)$$

by

$$\langle T\boldsymbol\psi_\alpha(x)\bar{\boldsymbol\psi}_\beta(y)\rangle = i\hbar c S_{\alpha\beta}(x,y). \tag{25}$$

We consider here the interacting fields and not the free fields as in Ref.[10] p.272. Therefore, $S(x,y)$ is the full electron Green function.

To define the current we follow Schwinger's prescription [51]. Let z be a four-vector with $z^0 > 0$, the four-current is

$$J^\mu(x) = -ie\hbar c\mathrm{tr}\big[\gamma^\mu S(x,x)\big], \tag{26}$$

where $S(x,x)$ is defined by the symmetric limit

$$S(x,x) = \frac{1}{2}\lim_{\mathbf{z}\to 0}\lim_{z^0\to 0^+}\big(S(x+z,x-z) \\ +S(x-z,x+z)\big). \qquad (27)$$

More precisely, we start from a non-zero 3-vector $\mathbf{z}$ (i.e. $|\mathbf{z}|\neq 0$) and a positive time z^0. We first make $z^0\to 0$ and then $|\mathbf{z}|\to 0$. Because of the symmetric limit we obtain the same result with $z^0<0$ (just take $z\to -z$ in the above definition)

$$S(x,x) = \frac{1}{2}\lim_{\mathbf{z}\to 0}\lim_{z^0\to 0^-}\big(S(x+z,x-z) \\ +S(x-z,x+z)\big).$$

From definition (25), for $z^0>0$ we have

$$S_{\alpha\beta}(x+z,x-z) = -\frac{i}{\hbar c}\langle \boldsymbol{\psi}_\alpha(x+z)\bar{\boldsymbol{\psi}}_\beta(x-z)\rangle, \qquad (28)$$

$$S_{\alpha\beta}(x-z,x+z) = \frac{i}{\hbar c}\langle \bar{\boldsymbol{\psi}}_\beta(x+z)\boldsymbol{\psi}_\alpha(x-z)\rangle. \qquad (29)$$

When we compare this with the definition of the bilocal current field operator, we obtain

$$J^\mu(x) = \lim_{\mathbf{z}\to 0}\lim_{z^0\to 0^+}\langle \mathbf{J}^\mu(x+z,x-z)\rangle.$$

On the other hand, if $z^0<0$

$$J^\mu(x) = \lim_{\mathbf{z}\to 0}\lim_{z^0\to 0^-}\langle \mathbf{J}^\mu(x-z,x+z)\rangle.$$

Thus, when the definition of $J^\mu(x)$ is written in terms of the bilocal operator, the first argument of the bilocal operator is always later than x. It is not clear that the current defined by Eq.(26) is real, as a measurable quantity should be.

Now we prove that the current is real and transforms properly under charge conjugation.

6.2 Self-adjointness

Using the definition $\bar{\boldsymbol{\psi}} = \boldsymbol{\psi}^\dagger\gamma^0$, the bilocal operator can be rewritten

$$\mathbf{J}^\mu(x,y) = \frac{e}{2}\big[\boldsymbol{\psi}(x)^\dagger\gamma^0\gamma^\mu\boldsymbol{\psi}(y) - \boldsymbol{\psi}(x)(\gamma^0\gamma^\mu)^T\boldsymbol{\psi}(y)^\dagger\big],$$

where A^T is the transpose of A. From the identity $(\gamma^0\gamma^\mu)^\dagger = \gamma^0\gamma^\mu$ (Ref.[10] p.693) we obtain immediately

$$\mathbf{J}^\mu(x,y)^\dagger = \mathbf{J}^\mu(y,x). \qquad (30)$$

Now we can make an important remark concerning the bilocal current. If x and y are separated by a space-like interval (i.e. if $(x-y)\cdot(x-y)<0$), then $\bar{\boldsymbol{\psi}}(x)$ and $\boldsymbol{\psi}(y)$ anticommute. This property is called causality and means that two events that are too far to be linked by a light-ray are independent. Causality is a basis of all axiomatic approaches to quantum field theory [53].

Thus, if $(x-y)\cdot(x-y)<0$ we have

$$\mathbf{J}^\mu(x,y) = \frac{e}{2}\sum_{\alpha\beta}\gamma^\mu_{\alpha\beta}\big(-\boldsymbol{\psi}_\beta(y)\bar{\boldsymbol{\psi}}_\alpha(x) + \bar{\boldsymbol{\psi}}_\alpha(y)\boldsymbol{\psi}_\beta(x)\big) \\ = \mathbf{J}^\mu(y,x).$$

Combined with Eq.(30), this means that the bilocal current is self-adjoint for space-like separated points.

The symmetric definition (27) of the current in terms of the Green function was chosen such that first $z^0\to 0^+$ with $|\mathbf{z}|\neq 0$. This is to ensure that, for z^0 small enough (i.e. $z^0<|\mathbf{z}|$), the interval z becomes space-like and the current is real.

Moreover, if x and y are space-like separated, we can rewrite the bilocal current in terms of field commutators:

$$\mathbf{J}^\mu(x,y) = \frac{\mathbf{J}^\mu(x,y)+\mathbf{J}^\mu(y,x)}{2} \\ = \frac{e}{4}\sum_{\alpha\beta}\gamma^\mu_{\alpha\beta}\big([\bar{\boldsymbol{\psi}}_\alpha(x),\boldsymbol{\psi}_\beta(y)] + [\bar{\boldsymbol{\psi}}_\alpha(y),\boldsymbol{\psi}_\beta(x)]\big). \qquad (31)$$

The fact that the current $J^\mu(x)$ defined by Eq.(26) is real can also be obtained directly from a property of $S(x,y)$. If z is a space-like vector, causality ensures that

$$\big(\psi_\alpha(x+z)\psi^\dagger_\beta(x-z)\big)^\dagger = \psi_\beta(x-z)\psi^\dagger_\alpha(x+z) \\ = -\psi^\dagger_\alpha(x+z)\psi_\beta(x-z).$$

Now Eq.(28) gives

$$\langle\psi_\alpha(x+z)\psi^\dagger_\beta(x-z)\rangle = i\hbar c\sum_\lambda S_{\alpha\lambda}(x+z,x-z)\gamma^0_{\lambda\beta},$$

and Eq.(29) gives

$$\langle\psi^\dagger_\alpha(x+z)\psi_\beta(x-z)\rangle = -i\hbar c\sum_\mu S_{\beta\mu}(x+z,x-z)\gamma^0_{\mu\alpha}.$$

Thus, we obtain, for space-like z

$$S^\dagger(x+z,x-z) = -\gamma^0 S(x-z,x+z)\gamma^0. \qquad (32)$$

Here the dagger operator acts only on the spin-variables: $S^\dagger{}_{\alpha\beta}(x+z,x-z) = S^*_{\beta\alpha}(x+z,x-z)$. From Eq.(32) and the identity $\gamma^0\gamma^\mu\gamma^0 = \gamma^{\mu\dagger}$ we can check that $J^\mu(x)^* = J^\mu(x)$.

6.3 Charge conjugation

Let $\mathcal{C}$ be the charge conjugation operator acting on fields and $C = i\gamma^2\gamma^0$ the charge conjugation matrix. We have the identities ([10] p.152[3] and [54] p.70)

$$\mathcal{C}\boldsymbol{\psi}(x;e)\mathcal{C}^\dagger = C\bar{\boldsymbol{\psi}}(x;-e), \\ \mathcal{C}\bar{\boldsymbol{\psi}}(x;e)\mathcal{C}^\dagger = \boldsymbol{\psi}(x;-e)C.$$

[3] Their misprinted definition of C is corrected on page 693 of [10].

In the presence of an external field, the sign of the charge in the field operators is reversed under charge conjugation ([15] p.19).

The action of the charge operator on the bilocal current is

$$\mathcal{C}\mathbf{J}^\mu(x,y;e)\mathcal{C}^\dagger = \frac{e}{2}\big(\boldsymbol{\psi}(x;-e)C\gamma^\mu C\bar{\boldsymbol{\psi}}(y;-e) \\ -\bar{\boldsymbol{\psi}}(x;-e)(C\gamma^\mu C)^T\boldsymbol{\psi}(y;-e)\big).$$

From the identities ([10] p.693) $C\gamma^\mu C^\dagger = -\gamma^{\mu T}$ and $C^T = C^\dagger = C^{-1} = -C$ we obtain

$$\mathcal{C}\mathbf{J}^\mu(x,y;e)\mathcal{C}^\dagger = -\mathbf{J}^\mu(x,y;-e).$$

Therefore, the sign of the bilocal current is reversed under charge conjugation.

Now we investigate the behaviour of the current under charge conjugation, when it is defined in terms of the Green function. To do this, we need an equation for the Green function of the charge conjugated problem. We obtain it by transposing both sides of Eq.(19). This transposition is meant for the 4x4 matrices only: ${S^T}_{ss'}(x,y) = S_{s's}(x,y)$. Then we multiply both sides of the equation by the charge conjugation matrix C on the left and C^{-1} on the right. We use the property $C\gamma^{\mu T}C^{-1} = -\gamma^\mu$ and we obtain

$$\delta(x-y) = (i\hbar c\gamma\cdot\partial - mc^2 + e\gamma\cdot A(x))S^c(x,y) \\ +ie\hbar c\gamma_\mu\frac{\delta S^c(x,y)}{\delta j_\mu(x)},$$

where we have defined $S^c(x,y) = CS(y,x)^T C^{-1}$. If we compare this equation with the second basic equation (15), we see that they become identical if we make the transformation $e\to -e$, or equivalently $A^\mu \to -A^\mu$ and $j^\mu \to -j^\mu$. But, using Eq.(12) and the charge conjugation property of the induced current $J^\mu(x)$, we see that $j^\mu\to -j^\mu$ implies $A^\mu\to -A^\mu$. In other words, we have the symmetry

$$S^c(x,y) = S(x,y;-j) = CS(y,x;j)^T C^{-1}.$$

Therefore

$$\begin{aligned}\mathrm{tr}[\gamma^\mu S(x-z,x+z;j)] &= \mathrm{tr}[C\gamma^\mu C^{-1}S(x+z,x-z;-j)^T] \\ &= -\mathrm{tr}[\gamma^{\mu T}S(x+z,x-z;-j)^T] \\ &= -\mathrm{tr}[\gamma^\mu S(x+z,x-z;-j)].\end{aligned}$$

And the current can be rewritten as the antisymmetric limit

$$S(x,x;j) = \frac{1}{2}\lim_{\mathbf{z}\to 0}\lim_{z^0\to 0^+}\big(S(x+z,x-z;j) \\ -S(x+z,x-z;-j)\big).$$

This equation shows that the current is an odd function of the external current: $J_\mu(x;j) = -J_\mu(x;-j)$. Changing the sign of j amounts to changing the sign of A, and we can also write $J_\mu(x;A) = -J_\mu(x;-A)$: the current is an odd function of A. Now, since the electron loops of QED are generated by the current, and since the x in $S(x,x;A)$ is linked to another photon propagator, we see that the electron loops with an odd number of photon lines are zero. This is a version of Furry's theorem for QED with external field. The relevance of charge conjugation for the QED current was already noticed by Kramers in 1937 [55]. Related results can be found in Ref.[7], sections 4.2, 9.4 and 15.1.

6.4 Charge conservation

Charge conservation is described by $\partial_\mu J^\mu(x) = 0$. How does this translate for our bilocal operator? We want a real and symmetric expression for $\partial_\mu J^\mu(x) = 0$. Why symmetric? If a bilocal operator $F(x,y)$ is antisymmetric (i.e. $F(x,y) = -F(y,x)$), then its value at $x=y$ is zero, except at divergences, such as $(x-y)/(x-y)^2$. Since the regular part of $F(x,y)$ is zero on the diagonal, the use of a symmetric operator eliminates the antisymmetric divergences without changing the value of the regular operator on the diagonal.

An obvious candidate is $(\partial^x_\mu + \partial^y_\mu)\mathbf{J}^\mu(x,y)/2$. From the field equations [7] p.195

$$(i\hbar c\partial_\mu - eA_\mu)\gamma^\mu\boldsymbol{\psi} - mc^2\boldsymbol{\psi} = 0,$$
$$(i\hbar c\partial_\mu + eA_\mu)\bar{\boldsymbol{\psi}}\gamma^\mu + mc^2\bar{\boldsymbol{\psi}} = 0,$$

we check that $(\partial^x_\mu + \partial^y_\mu)\mathbf{J}^\mu(x,y) = 0$.

In terms of the electron Green function, this becomes

$$-\frac{ie\hbar c}{4}\partial^x_\mu \mathrm{tr}[\gamma^\mu S(x+z,x-z) + \gamma^\mu S(x+z,x-z)],$$

where we have used $\partial^{x+z}_\mu + \partial^{x-z}_\mu = \partial^x_\mu$ and where z is space-like. Charge conservation can then be directly verified from Eqs.(15) and (19).

6.5 Mass-reversal symmetry

The concept of mass-reversal symmetry was discussed by Peaslee [56] and Tiomno [57]. For massless particles, mass-reversal symmetry is known as chiral symmetry.

The mass-reversed fermion field $\boldsymbol{\psi}(-m)$ satisfies the same equations as the usual fermion field $\boldsymbol{\psi}(m)$, but with a reversed mass:

$$(i\hbar c\gamma\cdot\partial - e\gamma\cdot A + mc^2)\boldsymbol{\psi}(-m) = 0.$$

Since $\gamma^\mu\gamma_5 = -\gamma_5\gamma^\mu$ we can take $\boldsymbol{\psi}(x;-m) = \gamma^5\boldsymbol{\psi}(x;m)$ (up to a phase factor η_m which disappears in the Green function). To see how the Green function transforms under mass reversal, we multiply both sides of Eq.(15) by γ_5 on the left and on the right. The commutation rules between γ_5 and γ^μ yield

$$\delta(x-y) = (-i\hbar c\gamma\cdot\partial - mc^2 + e\gamma\cdot A(x))\gamma_5 S(x,y;m)\gamma_5 \\ +ie\hbar c\gamma_\mu\gamma_5\frac{\delta S(x,y;m)}{\delta j_\mu(x)}\gamma_5, \quad (33)$$

which is the equation for $-S(x,y;-m)$. Therefore,

$$S(x,y;-m) = -\gamma_5 S(x,y;m)\gamma_5,$$

and the mass-reversed current is

$$\begin{aligned}J^\mu(x;-m) &= -ie\hbar c \mathrm{tr}[\gamma^\mu S(x,x;-m)]\\ &= ie\hbar c\mathrm{tr}[\gamma_5\gamma^\mu\gamma_5 S(x,x;m)] = J^\mu(x;m).\end{aligned}$$

Thus, the induced current is even under mass-conjugation, and the vector potential also, as expected [57].

In other words, the induced current is an even function of m.

6.6 Gauge invariance

The Lagrangian of QED is invariant under the gauge transformations ([10] p.64)

$$\begin{aligned}\boldsymbol{\psi}^\Lambda(x) &= \mathrm{e}^{-ie\Lambda(x)}\boldsymbol{\psi}(x),\\ \mathbf{A}^\Lambda_\mu(x) &= \mathbf{A}_\mu(x) + \hbar c\partial_\mu\Lambda(x).\end{aligned}$$

After a gauge transformation, the bilocal current for space-like separated points Eq.(31) becomes

$$\begin{aligned}\mathbf{J}^\Lambda(x,y) &= \cos(e\Lambda(x) - e\Lambda(y))\mathbf{J}(x,y)\\ &+\frac{ie}{4}\sin(e\Lambda(x)-e\Lambda(y))\\ \times\sum_{\alpha\beta}&\gamma^\mu_{\alpha\beta}\big([\bar{\boldsymbol{\psi}}_\alpha(x),\boldsymbol{\psi}_\beta(y)] - [\bar{\boldsymbol{\psi}}_\alpha(y),\boldsymbol{\psi}_\beta(x)]\big).\end{aligned} \quad (34)$$

In terms of the electron Green function, we must consider the transformed Green functions

$$S^\Lambda(x,y) = \mathrm{e}^{-ie\Lambda(x)+ie\Lambda(y)}S(x,y). \quad (35)$$

Hence,

$$\begin{aligned}&S^\Lambda(x+z,x-z) + S^\Lambda(x-z,x+z) =\\ &\cos(e\Lambda(x)-e\Lambda(y))\big(S(x+z,x-z)+S(x-z,x+z)\big)\\ &-i\sin(e\Lambda(x)-e\Lambda(y))\big(S(x+z,x-z)-S(x-z,x+z)\big).\end{aligned}$$

A priori, the singularities of the Green function are not weak enough to make the second term tend to zero when $z\to 0$. Therefore, our definition of the current is not obviously gauge invariant.

This problem can be solved by using a Green function multiplied by Dirac's phase [30]

$$\begin{aligned}S'(x+z,x-z) = \exp\big[\frac{ie}{\hbar c}\int_{-1}^1 \mathrm{d}\lambda\, z^\mu A_\mu(x+\lambda z)\big]\\ \times S(x+z,x-z).\end{aligned}$$

A generalization of this phase was used recently for the renormalization of infrared divergences in QED [58,59].

The modified Green function $S'(x+z,x-z)$ is gauge invariant because, according to Eq.(58) in the appendix,

$$\int_{-1}^1 \mathrm{d}\lambda\partial_\mu\Lambda(x+\lambda z)z^\mu = \Lambda(x+z) - \Lambda(x-z).$$

The right hand side compensates for the gauge transformation phase in Eq.(35), so that ${S'}^\Lambda(x,y) = S'(x,y)$.

If we substitute $S'(x+z,x-z)$ for $S(x+z,x-z)$ in the definition of the current (26), we obtain a current which is real, odd under charge conjugation and gauge invariant, but charge conservation becomes doubtful. Thus, we stick to the original definition Eq.(26).

7 Renormalization

The renormalization of QED in the presence of an external field does not seem to have been treated in detail. Several attempts exist [1,40,52,60,61], but none of them was fully completed. With respect to the usual renormalization of QED in the vacuum, we have three additional difficulties: (i) translation invariance is lost so that counterterms can depend on the position (for the current), (ii) we work in the space of positions and not in the space of momenta, (iii) tadpoles are present (for strong field QED).

Our first step is to determine Dyson's relations, which are non perturbative expressions that link renormalized and non renormalized propagators and fields. Then we give a non perturbative relation between the renormalized potential and the renormalized vacuum polarization. Finally, we describe the renormalization rules that enable us to obtain finite quantities for any strong field or SCF Feynman diagram.

7.1 Dyson relations for strong field QED

The first point is to determine the Dyson relations between renormalized and unrenormalized propagators. This relation is given by Sterman[62] for scalar fields and we derive it for QED. If we denote the renormalized quantities with a overlined symbol, in the presence of an external potential $a_\mu(x)$, Dyson's relations are [63]

$$\begin{aligned}S(x,y;a;e_0,m_0) &= Z_2\bar{S}(x,y;\sqrt{Z_3}a;e,m),\\ D(x,y;a;e_0,m_0) &= Z_3\bar{D}(x,y;\sqrt{Z_3}a;e,m),\\ A(x;a;e_0,m_0) &= \sqrt{Z_3}\bar{A}(x;\sqrt{Z_3}a;e,m).\end{aligned}$$

To derive these relations, we expand S over a:

$$S(x,y;a) = S(x,y;0) + \int \mathrm{d}z_1 \frac{\delta S(x,y;0)}{\delta a_\mu(z_1)}a_\mu(z_1) + \cdots$$

The Dyson relation for the vacuum propagator $S(x,y;0)$ is $S(x,y;0) = Z_2\bar{S}(x,y;0)$. For the other terms, the functional derivative with respect to the external potential brings a factor $A(x)$ in the path integral. Therefore,

$$\frac{\delta^n S(x,y;0)}{\delta a^n} = Z_2 Z_3^{n/2}\frac{\delta^n\bar{S}(x,y;0)}{\delta a^n}.$$

Hence,

$$\begin{aligned}S(x,y;a) &= Z_2\bar{S}(x,y;0) + Z_2\int \mathrm{d}z_1\frac{\delta\bar{S}(x,y;0)}{\delta a_\mu(z_1)}\sqrt{Z_3}a_\mu(z_1)\\ &+\cdots\\ &= Z_2\bar{S}(x,y;\sqrt{Z_3}a).\end{aligned}$$

Or, more precisely,

$$S(x,y;a;e_0,m_0) = Z_2\bar{S}(x,y;\sqrt{Z_3}a;e,m).$$

Because of this relation, we also define the renormalized external potential as $\bar{a}_\mu(x) = \sqrt{Z_3}a_\mu(x)$. It may seem strange that the external potential $a_\mu(x)$ and the full potential $A_\mu(x)$ are not renormalized with the same formula. However, we must recall that $a_\mu(x)$ was defined in terms of an external current $j_\mu(x)$, which was used to generate Green functions by functional derivatives. The bare Green functions are generated by $A^\mu(x)j_\mu(x)$ and the renormalized ones by $\bar{A}^\mu(x)\bar{\jmath}_\mu(x)$. Since we want these two terms to be equal, the relation $\sqrt{Z_3}\bar{A}_\mu(x) = A_\mu(x)$ implies $\bar{\jmath}_\mu(x) = \sqrt{Z_3}j_\mu(x)$ (see Refs.[62] p. 297, [19] p.288 and especially [60] p.34). More physically, we can say that an external field is made by preparing a certain density of (charged) matter $n(\mathbf{x})$. Before renormalization, this density corresponds to a charge density $j^0(\mathbf{x}) = e_0 n(\mathbf{x})$. Renormalization modifies the charge, but not the density of matter. Thus the renormalized charge density is now $\bar{\jmath}^0(\mathbf{x}) = en(\mathbf{x}) = \sqrt{Z_3}j^0(\mathbf{x})$, and the corresponding relation for the external potential is again $\bar{a}^\mu(x) = \sqrt{Z_3}a^\mu(x)$. We insist on this point because external and full potentials are sometimes renormalized with the same equation in the literature [1,2]. Finally, the relation between the renormalized and nonrenormalized functions is

$$\bar{S}(x,y;\bar{a};e,m) = \frac{1}{Z_2}S(x,y;\frac{\bar{a}}{\sqrt{Z_3}};e_0,m_0), \quad (36)$$

$$\bar{D}(x,y;\bar{a};e,m) = \frac{1}{Z_3}D(x,y;\frac{\bar{a}}{\sqrt{Z_3}};e_0,m_0), \quad (37)$$

$$\bar{A}(x;\bar{a};e,m) = \frac{1}{\sqrt{Z_3}}A(x;\frac{\bar{a}}{\sqrt{Z_3}};e_0,m_0). \quad (38)$$

In terms of the renormalized propagators, the functional equation becomes (using $e_0 = e/\sqrt{Z_3}$ and $m_0 = m + \delta m$)

$$(i\hbar c\gamma\cdot\partial - mc^2 - \delta mc^2 - e\gamma\cdot\bar{A}(x))Z_2\bar{S}(x,y) = \delta(x-y)$$
$$+ie\hbar cZ_2\gamma^\mu\int ds\frac{\delta\bar{S}(x,y)}{\delta\bar{a}_\lambda(s)}D^0_{\lambda\mu}(s,x).$$

and

$$\sqrt{Z_3}\bar{A}_\mu(x) = \bar{a}_\mu(x)/\sqrt{Z_3} - i\frac{e}{\sqrt{Z_3}}\hbar cZ_2$$
$$\times\int dsD^0_{\mu\nu}(x,y)\mathrm{tr}[\gamma^\nu\bar{S}(y,y)],$$

or

$$\bar{A}_\mu(x) = \frac{\bar{a}_\mu(x)}{Z_3} - i\frac{eZ_2}{Z_3}\hbar c\int dsD^0_{\mu\nu}(x,y)\mathrm{tr}[\gamma^\nu\bar{S}(y,y)], \quad (39)$$

7.2 Dyson relations for SCF-QED

Notice that, in the case of SCF-QED, Dyson's relations are much simpler, because $e_0A = e\bar{A}$:

$$S(x,y;e_0A;e_0,m_0) = Z_2\bar{S}(x,y;e\bar{A};e,m), \quad (40)$$

$$D(x,y;e_0A;e_0,m_0) = Z_3\bar{D}(x,y;e\bar{A};e,m), \quad (41)$$

$$A(x;e_0A;e_0,m_0) = \sqrt{Z_3}\bar{A}(x;e\bar{A};e,m). \quad (42)$$

The main difference between vacuum QED and SCF-QED is the renormalization of the induced vector potential. The total bare potential is

$$A_\mu(x) = a_\mu(x) - ie_0\hbar c\int \mathrm{d}sD^0_{\mu\nu}(x,s)\mathrm{tr}[\gamma^\nu S(s,s)], \quad (43)$$

and the problem is the renormalization of $\mathrm{tr}[\gamma^\nu S(s,s)]$. To indicate that the electron Green function is calculated in the presence of the full potential $A_\mu(x)$ we denote it by $S(x,y;A)$.

Dyson's relation (40) gives us

$$\mathrm{tr}[\gamma^\nu S(s,s;A)] = Z_2\mathrm{tr}[\gamma^\nu\bar{S}(s,s;\bar{A})].$$

We rewrite the right hand side using Eq.(56):

$$\mathrm{tr}[\gamma^\nu\bar{S}(s,s;\bar{A})] = \mathrm{tr}[\gamma^\nu\bar{S}(s,s;0)] + \int_0^1\mathrm{d}\lambda\int\mathrm{d}x$$
$$\times\mathrm{tr}\Big[\gamma^\nu\frac{\delta\bar{S}(s,s;\lambda\bar{A})}{\delta\bar{A}^\mu(x)}\Big]\bar{A}^\mu(x).$$

Now $-ie\hbar c\mathrm{tr}[\gamma^\nu\bar{S}(s,s;0)]$ is the current in the presence of a zero external field $\bar{A} = 0$. Thus, it is the vacuum current, which is zero (more precisely, $\mathrm{tr}[\gamma^\nu\bar{S}(s,s;0)]$ is made of fermion loops with one external photon line, which are zero by Furry's theorem).

The second term is also well known from the renormalization of the vacuum polarization (see e.g. Eq.(22) of Ref.[64])

$$-ie\hbar cZ_2\mathrm{tr}\Big[\gamma_\nu\frac{\delta\bar{S}(s,s;\lambda\bar{A})}{\delta\bar{A}^\mu(y)}\Big] =$$
$$\bar{\Pi}_{\nu\mu}(s,y;\lambda\bar{A}) + (Z_3-1){D^0}^{-1}_{\mu\nu}(s,y). \quad (44)$$

Thus, we obtain

$$-ie\hbar cZ_2\int\mathrm{d}y\mathrm{tr}[\gamma_\nu\bar{S}(s,s;\bar{A})] =$$
$$\int_0^1\mathrm{d}\lambda\int\mathrm{d}y\bar{\Pi}_{\nu\mu}(s,y;\lambda\bar{A})\bar{A}^\mu(y)$$
$$+(Z_3-1)\int\mathrm{d}y{D^0}^{-1}_{\mu\nu}(s,y)\bar{A}^\mu(y).$$

After this reorganization of the induced current, Eq.(43) becomes

$$A_\mu(x) = a_\mu(x) + \frac{Z_3-1}{\sqrt{Z_3}}\bar{A}_\mu(x)$$
$$+\frac{1}{\sqrt{Z_3}}\int\mathrm{d}s\mathrm{d}y\mathrm{d}\lambda D^0_{\mu\nu}(x,s)\bar{\Pi}^{\nu\rho}(s,y;\lambda\bar{A})\bar{A}_\rho(y).$$

Now we know that the relation between the bare and renormalized potentials is $A_\mu(x) = \sqrt{Z_3}\bar{A}_\mu(x)$. This yields

$$\bar{A}_\mu(x) = \bar{a}_\mu(x) + \int_0^1 \mathrm{d}\lambda \int \mathrm{d}s\mathrm{d}y D^0_{\mu\nu}(x,s) \times \bar{\Pi}^{\nu\rho}(s,y;\lambda\bar{A})\bar{A}_\rho(y), \qquad (45)$$

where the true external potential $a_\mu(x)$ is renormalized by $\bar{a}_\mu(x) = \sqrt{Z_3}a_\mu(x)$.

It should be noticed that, in all our manipulations, the renormalization factors Z_2, Z_3 and δm where taken at $\bar{A}_\mu = 0$. In other words, the renormalization factors of QED with external field are the same as the renormalization factors of vacuum QED. More precisely, the vacuum renormalization factors removes the divergences of QED in external field. However, renormalization conditions may introduce finite differences between the renormalization factors of vacuum QED and QED in an external field.

From Eq. (45), it can be checked that $\bar{\Pi}$ is indeed the renormalized vacuum polarization. If we write $X_\mu(x)$ for the second term on the right hand side of Eq. (45) we find

$$\begin{aligned}\bar{D}_{\mu\nu}(x,y) &= \frac{\delta\bar{A}_\mu(x)}{\delta\bar{\jmath}^\nu(y)} = \frac{\delta\bar{a}_\mu(x)}{\delta\bar{\jmath}^\nu(y)} + \frac{\delta X_\mu(x)}{\delta\bar{\jmath}^\nu(y)} \\ &= D^0_{\mu\nu}(x,y) + \int \mathrm{d}z \frac{\delta X_\mu(x)}{\delta\bar{A}_\rho(z)}\frac{\delta\bar{A}_\rho(x)}{\delta\bar{\jmath}^\nu(y)} \\ &= D^0_{\mu\nu}(x,y) + \int \mathrm{d}s \int \mathrm{d}z D^0_{\mu\sigma}(x,s) \\ &\quad \times \bar{\Pi}^{\sigma\rho}(s,z)\bar{D}_{\rho\nu}(z,y),\end{aligned} \qquad (46)$$

where we used

$$\frac{\delta X_\mu(x)}{\delta\bar{A}_\rho(z)} = \int \mathrm{d}s D^0_{\mu\sigma}(x,s)\bar{\Pi}^{\sigma\rho}(s,z).$$

This last equation is proved by using Eq.(44) to rewrite $X_\mu(x)$ in terms of a functional derivative of $\bar{S}$, and by using Eq.(59) to carry out the integral over λ.

Equation (46) expresses the usual relation between the photon Green function and the vacuum polarization.

8 Weak field renormalization

In this section, we give the renormalization rules of vacuum QED, which are valid for weak field renormalization. We repeat some of the rules given in [64] for completeness, and because we work now in the direct space (and not in the Fourier space). The first set of rules is equivalent to Zimmermann's forest formula for the removal of subdivergences. The second set of rules subtracts the superficial divergences.

8.1 Removal of subdivergences

If Γ is a Feynman diagram, and $U(\Gamma)$ is the regularized value of the diagram as a function of external momenta and masses and regularization parameters, then the value of the diagram with subdivergences subtracted is $\bar{R}(\Gamma)$ defined by [10,65,66]

$$\bar{R}(\Gamma) = U(\Gamma) + \sum_{\{\gamma i,\gamma' i',\dots\}} C_i(\gamma)C_{i'}(\gamma')\cdots \times U(\Gamma/\{\gamma_{(i)},\gamma'_{(i')},\dots\}). \qquad (47)$$

In this expression, the sum runs over all sets of disjoint renormalization parts of Γ. A renormalization part γ of Γ is a one-particle irreducible (1PI) subgraph of Γ, different from Γ itself, such that γ has two or three (amputated) external lines. A diagram is 1PI when it is connected and cannot be disconnected by cutting through any of its internal lines. Two renormalization parts γ and γ' are disjoint if they have no vertex in common.

There are three types of renormalization part: self-energy 1PI diagrams ($\gamma =$), vacuum polarization 1PI diagrams ($\gamma =$) and reduced vertex 1PI diagrams ($\gamma =$). The index (i) depends on the type of the renormalization part. If $\gamma =$ then i is 0 or 2, if $\gamma =$ then i is 3, if $\gamma =$ then i is 1.

Finally let us define $\Gamma/\{\gamma_{(i)},\gamma'_{(i')},\dots\}$. We start by the definition of $\Gamma/\gamma_{(i)}$. It varies with the type of renormalization part and with the index (i).

If $\gamma =$ is a self-energy 1PI diagram, then i can be 0 or 2. In the complete diagram Γ, the self-energy subdiagram $\gamma =$ is a part of an electron propagator . In the term $\Gamma/\gamma_{(0)}$, the propagator diagram is replaced by a product of two free propagators , in the term $\Gamma/\gamma_{(2)}$ the propagator diagram is replaced by a single free propagator . In the direct space, the product of two propagators means $= \int \mathrm{d}z S^0(x,z)S^0(z,y)$.

If $\gamma =$ is a vacuum polarization 1PI diagram, then i is 3. In the complete diagram Γ, the vacuum polarization $\gamma =$ is a part of a photon propagator . In the term $\Gamma/\gamma_{(3)}$, the propagator diagram is replaced by a single free propagator .

If $\gamma =$ is a reduced vertex 1PI diagram, then i is 1. In the complete diagram Γ, the reduced vertex $\gamma =$ is a part of a vertex diagram . In the term $\Gamma/\gamma_{(1)}$, the vertex diagram is replaced by a free vertex . The terms $\Gamma/\{\gamma_{(i)},\gamma'_{(i')},\dots\}$ are then defined recursively. For instance, to define $\Gamma'' = \Gamma/\{\gamma_{(i)},\gamma'_{(i')}\}$, we first put $\Gamma' = \Gamma/\gamma_{(i)}$, so that $\Gamma'' = \Gamma'/\gamma'_{(i')}$.

We insist here on the fact that the removal of subdivergences given by Eq.(47) is valid for any (connected or disconnected) Feynman diagram. It is the expression, for each diagram, of Dyson's relations. The reason why the counterterms $C_1(\gamma)$ are required, although $Z_2 = Z_1$, is explained in [64].

In Eq.(47) all counterterms $C_i(\gamma)$ are assumed to be known from the renormalization of the superficial divergence of the subdiagram γ.

8.2 Superficial divergences

In the second step, we determine the counterterms $C_i(\Gamma)$ of the divergent graph Γ. If the diagram Γ is not 1PI, the counterterms $C_i(\Gamma)$ are zero. If Γ is 1PI, we must distinguish three cases. A self-energy diagram is linearly divergent, thus we must remove two terms. From Lorentz covariance, we can write the renormalized value of the diagram Γ as

$$R(\Gamma;x,y) = \bar{R}(\Gamma;x,y) + C_0(\Gamma)c^2\delta(x-y) + C_2(\Gamma){S^0}^{-1}(x,y). \quad (48)$$

A vacuum polarization diagram is quadratically divergent, thus we should have to remove three terms. However, Lorentz covariance and the Ward identities for the photon propagators cancel the first two counterterms, and the renormalized value of Γ is

$$R_{\mu\nu}(\Gamma;x,y) = \bar{R}_{\mu\nu}(\Gamma;x,y) + C_3(\Gamma){D^0}^{-1}_{\mu\nu}(x,y). \quad (49)$$

Finally, a reduced vertex diagram is logarithmically divergent, and its renormalized value is

$$R(\Gamma;x,y;\lambda,z) = \bar{R}(\Gamma;x,y;\lambda,z) + C_1(\Gamma)\gamma^\lambda\delta(x-y)\delta(y-z). \quad (50)$$

The infinite constants $C_i(\Gamma)$ are determined from the renormalization conditions.

It was noticed in Ref.[64] that

$$C_2(\Gamma) = -\sum_j C_1(\Gamma_j), \quad (51)$$

where Γ_j is the vertex diagram obtained by branching a photon line on the j-th free electron propagator of the self-energy diagram Γ.

Since the renormalization of weak field QED is exactly the same as the renormalization of vacuum QED, we refer the reader to Refs.[10] and [64] for examples.

9 Strong field renormalization

For the strong field renormalization, the removal of subdivergences is again given by Eq.(47), which is a purely algebraic identity related to the Hopf algebra structure of renormalization [66], [67].

According to the general strategy of renormalization, the non-renormalized diagrams are evaluated in terms of renormalized electron propagators. In vacuum QED, the mass m used in the free electron propagator is finite, it is not the infinite mass m_0 of the bare free electron propagator. Similarly, the charge and the potential used in the basic electron propagator of strong field QED are finite. In other words we take

$$S^{\bar{a}} = \big(i\hbar c\gamma\cdot\partial - mc^2 - e\gamma\cdot\bar{a}(x)\big)^{-1}.$$

Therefore, the external potential diagram means now

$U($$) = \bar{a}_\mu(x).$

To remove the superficial divergences we start by examining all possible superficially divergent diagrams (Chapter 8 of Ref.[10]). A priori, the superficially divergent 1PI diagrams are the self-energy and vertex diagrams, and the 1PI diagrams with one external photon (tadpole), two external photons (vacuum polarization), three external photons and four external photons (scattering of light by light).

9.1 Self-energy and vertex diagrams

To renormalize self-energy diagrams, we take the example of the one-loop diagram, and we expand the electron propagator using the Born expansion ([68] p.572)

$$S^{\bar{a}} = S^0 + eS^0\bar{a}\cdot\gamma S^0 + e^2 S^0\bar{a}\cdot\gamma S^0\bar{a}\cdot\gamma S^{\bar{a}}.$$

Thus

The strong field diagram on the left hand side is denoted Γ. The first diagram on the right hand side (denoted Γ') is a self-energy diagram of vacuum QED, the second diagram (denoted Γ'_1) is a vertex diagram of vacuum QED and the third diagram is finite by power counting. Using Eq.(48), the counterterms for Γ' are

$$R(\Gamma';x,y) = \bar{R}(\Gamma';x,y) + C_0(\Gamma')c^2\delta(x-y) + C_2(\Gamma'){S^0}^{-1}(x,y),$$

Using Eq.(50), the counterterm for the second diagram is

$$\int \mathrm{d}z R(\Gamma'_1;x,y;\lambda,z)\bar{a}_\lambda(z) = \int \mathrm{d}z \bar{R}(\Gamma'_1;x,y;\lambda,z)\bar{a}_\lambda(z) + C_1(\Gamma'_1)\gamma^\lambda\delta(x-y)\bar{a}_\lambda(y).$$

The total counterterms for the strong field diagram Γ are obtained by adding the counterterms for Γ' and Γ'_1. From the relation (51) between $C_1(\Gamma'_1)$ and $C_2(\Gamma')$ we obtain for the strong field self-energy diagrams

$$\begin{aligned} R(\Gamma;x,y) &= \bar{R}(\Gamma;x,y) + C_0(\Gamma)c^2\delta(x-y) \\ &\quad + C_2(\Gamma)({S^0}^{-1}(x,y) - e\gamma\cdot\bar{a}(y)\delta(x-y)) \\ &= \bar{R}(\Gamma;x,y) + C_0(\Gamma)c^2\delta(x-y) \\ &\quad + C_2(\Gamma){S^{\bar{a}}}^{-1}(x,y), \end{aligned} \quad (52)$$

where $C_0(\Gamma) = C_0(\Gamma')$ and $C_2(\Gamma) = C_2(\Gamma')$. Therefore, the strong field self-energy diagrams are renormalized with the same formula as the vacuum diagrams, the counterterms $C_0(\Gamma)$ and $C_1(\Gamma)$ are the same as for the vacuum case, the only difference is that the free propagator S^0 is replaced by the strong field propagator $S^{\bar{a}}$. More precisely, the counterterms are the same as those of vacuum QED if the renormalization conditions are compatible. Otherwise, their difference is finite.

For a general strong field self-energy diagram Γ, the proof is similar. The subdivergences are removed with formula (47). The same Born expansion is made. The first diagram Γ' is now the diagram Γ (with subtracted subdivergences) where all strong field electron propagators S^a are replaced by free electron propagators S^0, and Γ'_j is the diagram obtained by adding a photon line to the j-th electron line of Γ'. The remaining diagrams are finite by power counting. The superficial divergences of Γ' and Γ'_j are removed by the vacuum QED renormalization prescription, and we find Eq.(52) again.

For a strong field vertex diagram, we have a logarithmic divergence and the result is exactly the same as for vacuum QED:

$$R(\Gamma;x,y;\lambda,z) = \bar{R}(\Gamma;x,y;\lambda,z) + C_1(\Gamma)\gamma^\lambda\delta(x-y)\delta(y-z), \quad (53)$$

where $C_1(\Gamma)$ is the same as for vacuum QED, up to possible finite terms.

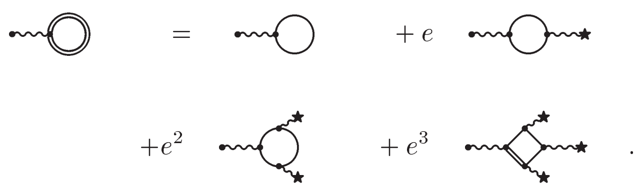
9.2 Vacuum polarization and tadpoles

We still have the subdivergence-subtracted 1PI diagrams with one, two, three or four external photon lines (and no external electron line). If the diagram has three or four external photon lines, then it is finite. To prove this, we expand the strong field electron propagator using the Born expansion. Then we are back to the case of vacuum QED, where we know that a 1PI diagram with three external photon lines is zero and a 1PI with four external photon lines is finite [10] (once subdivergences are subracted). A 1PI diagram with two external photon lines is a vacuum polarization diagram, which is renormalized as for vacuum QED with Eq.(49)

$$R_{\mu\nu}(\Gamma;x,y) = \bar{R}_{\mu\nu}(\Gamma;x,y) + C_3(\Gamma){D^0}^{-1}_{\mu\nu}(x,y), \quad (54)$$

where $C_3(\Gamma)$ has the same value as for the corresponding vacuum QED diagram, up to a possible finite quantity.

Now comes the diagram which is really typical of QED with external field: the tadpole. The induced vector potential $A^\mu(x) - \bar{a}^\mu(x)$ is given by the sum of all tadpole diagrams Γ. In vacuum QED, the induced vector potential is zero and the tadpoles do not intervene. For the case of QED with an external field, once subdivergences are removed, the tadpole has a cubic superficial divergence. But because of Furry's theorem (i.e. of the fact that the current is odd under charge conjugation), the only divergence that remains is the same as for vacuum polarization. We illustrate the general procedure with the simplest example (see also [23] and [60]). By using the Born expansion for $S^{\bar{a}}$ we obtain



Furry's theorem eliminates electron loops with one and three external photon lines (the first and the third diagrams on the right hand side). More precisely, since the induced current is an odd function of the external potential, the first and the third diagrams are not present on the right hand side. The last diagram is finite, because if we make a further Born expansion of $S^{\bar{a}}$ in it, we obtain a vacuum photon-photon scattering diagram, which is finite, plus a diagram with five external photon lines, which is finite by power counting. Therefore, the only divergent diagram on the rhs is the second one, which is a vacuum polarization diagram. We denote Γ'_1 this vacuum polarization diagram and Γ the strong field tadpole on the lhs.

We can write the bare potential as a sum of Feynman diagrams

$$A_\lambda(x) = a_\lambda(x) + \sum_\Gamma e_0^{|\Gamma|} U_\lambda(x;\Gamma),$$

where $|\Gamma|$ is the number of vertices of the tadpole diagram Γ. If Γ is the simple tadpole of the above example, our discussion shows that its value is

$$U_\lambda(x;\Gamma) = e\int \mathrm{d}y\mathrm{d}z D^0_{\lambda\mu}(x,y)\bar{R}^{\mu\nu}(\Gamma'_1;y,z)\bar{a}_\nu(z) + \text{finite terms}.$$

Now, we introduce the counterterm of Γ'_1 using Eq.(49), and we obtain the renormalized induced potential

$$R_\lambda(x;\Gamma) = e\int \mathrm{d}y\mathrm{d}z D^0_{\lambda\mu}(x,y)\bar{R}^{\mu\nu}(\Gamma'_1;y,z)\bar{a}_\nu(z) + eC_3(\Gamma'_1)\bar{a}_\lambda(x) + \text{finite terms}.$$

Similar equations can be found in [68] p. 552, [60,61].

Therefore, the renormalized value of the tadpole diagram Γ is

$$R_\lambda(\Gamma;x) = \bar{R}_\lambda(\Gamma;x) + eC_4(\Gamma)\bar{a}_\lambda(x),$$

where $C_4(\Gamma) = C_3(\Gamma'_1)$.

For a general tadpole diagram Γ, the result is similar. If $\bar{R}^\mu(\Gamma;x)$ is the value of Γ when all subdivergences are removed, the superficial divergence is removed with the counterterm

$$R^\mu(\Gamma;x) = \bar{R}^\mu(\Gamma;x) + eC_4(\Gamma)\bar{a}^\mu(x), \quad (55)$$

where

$$C_4(\Gamma) = \sum_i C_3(\Gamma_i).$$

In the last relation, Γ_i is the vacuum polarization obtained from the tadpole Γ by adding a photon line to the i-th electron line of Γ and by transforming all electron lines $S^{\bar{a}}$ into free electron lines S^0 in the resulting diagram. In other words, Γ_i is a vacuum QED vacuum polarization diagram, and $C_3(\Gamma_i)$ the corresponding counterterm.

After this analysis, we obtain the following addition to the description of the renormalization of vacuum QED. There is now a new index $i = 4$ and new 1PI diagrams γ which are tadpoles. In the term $\Gamma/\gamma_{(4)}$, the tadpole subdiagram $\gamma_{(4)}$ becomes an external potential line .

9.3 Summary

In this section, we summarize the renormalization rules for strong field QED. For any strong field Feynman diagram Γ, the subdivergences are subtracted by applying the forest formula (47). The only difference with the weak field case is that the vacuum electron propagators are replaced by strong field electron propagators.

When the subdivergences are subtracted, the superficial divergence is treated as follows. If Γ is not 1PI, the superficial counterterms are zero. In particular, if Γ contains tadpoles, then the counterterms $C_i(\Gamma)$ are zero. This is because a tadpole inside Γ can always be disconnected from Γ by cutting a photon line.

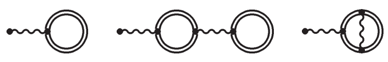
If Γ is 1PI, then the superficial divergence is removed using Eq.(52) for a self-energy and the indices are $i = 0$ and $i = 2$, Eq.(53) for a vertex and the index is $i = 1$, Eq.(54) for a vacuum polarization with the index $i = 3$ and Eq.(55) for a tadpole with the index $i = 4$.

Now we give a few example to illustrate the general rules.

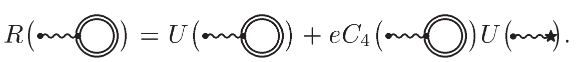
9.4 Example 1: the potential

As far as we know, the renormalization rules for strong field QED are new. Therefore, a number of examples are required to see how they work in practice.

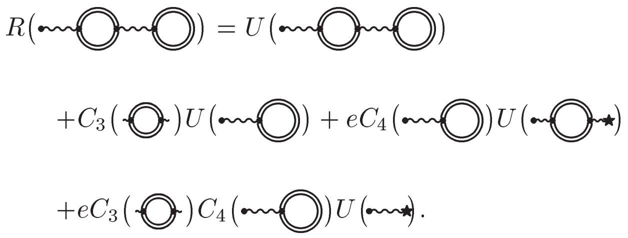
We start with the renormalization of the potential for the diagrams given in section 4.2.1. Up to order e^3, three diagrams must be calculated:



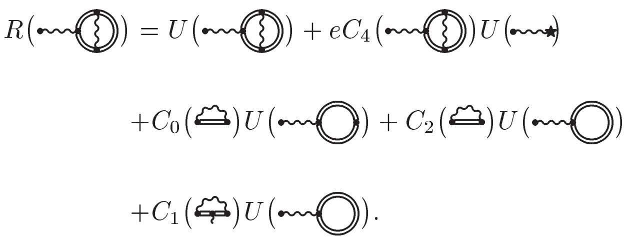
The first diagram has no subdivergence. It is renormalized with Eq.(55).



The second diagram is not 1PI. To renormalize it, we renormalize its two disjoint divergent subdiagrams: the vacuum polarization diagram on the left (index $i = 3$) and the tadpole on the right (index $i = 4$), and we multiply the renormalized diagrams. The general rule for a non 1PI diagram Γ is the following: write Γ as a product of 1PI diagrams, renormalized each 1PI diagram, and multiply the renormalized diagrams [18]. This rule is equivalent to the forest formula (47) for non 1PI diagrams, because renormalization parts belonging to different 1PI subdiagrams are disjoint.



For the last diagram, there are two subdivergences: on the right, a self-energy γ (indices $i = 0$ and $i = 2$) and on the left, a vertex γ' (index $i = 1$). Since these subdiagrams are not disjoint we obtain, adding the superficial divergence counterterm



To check these results, we make use of the renormalization factors.

$$Z_2 = 1 + \sum_{n=1}^{\infty} e^{2n} Z_{2,n},$$
$$Z_3 = 1 - \sum_{n=1}^{\infty} e^{2n} Z_{3,n},$$
$$\delta m = \sum_{n=1}^{\infty} e^{2n} \delta m_n.$$

The relation between the renormalization factors and the counterterms is as follows [64]. $Z_{2,n}$ is the sum of all $C_1(\Gamma)$ where Γ runs over the (1PI) vertex diagrams with n loops, $Z_{2,n}$ is also the sum of all $-C_2(\Gamma)$ where Γ runs over the (1PI) self-energy diagrams with n loops, $Z_{3,n}$ is the sum of all $C_3(\Gamma)$ where Γ runs over the (1PI) vacuum polarization diagrams with n loops, $Z_{3,n}$ is also the sum of all $C_4(\Gamma)$ where Γ runs over the (1PI) tadpole diagrams with n loops. Finally, $\delta m_n + \sum_{k=1}^{n-1} \delta m_k Z_{2,n-k}$ is the sum of all $C_0(\Gamma)$ where Γ runs over the (1PI) self-energy diagrams with n loops.

For our example, we have

$$Z_{2,1} = C_1(\;) = -C_2(\;),$$

$$Z_{3,1} = C_3(\;) = C_4(\;),$$

$$\delta m_1 = C_0(\;).$$

Thus, the renormalized induced potential up to two loops is

$$eR(\;) + e^3 R(\;) + e^3 R(\;) =$$

$$eU(\;) + e^3 U(\;) + e^3 U(\;)$$

$$+\big(e^2 Z_{3,1} + e^4(Z_{3,2} + Z_{3,1}^2)\big)U(\;)$$

$$+e^3 Z_{3,1} U(\;) + e^3 Z_{3,1} U(\;)$$

$$+e^3 \delta m_1 U(\;).$$

To check this result, we take Dyson's relation between renormalized and bare electron Green functions

$$Z_2(e,m)\bar S(x,y;e\bar a;e,m) = S(x,y;\frac{e\bar a}{Z_3};\frac{e}{\sqrt{Z_3}},m+\delta m).$$

We expand the right hand side

$$Z_2(e,m)\bar S(x,y;e\bar a;e,m) = U(\;) + e^2 Z_{3,1} U(\;)$$

$$+e^2\delta m_1 U(\;) + e^2 U(\;)$$

$$+e^2 U(\;) + \cdots$$

where the electron propagator is calculated in the field of the renormalized external potentials $\bar a(x)$. If we use this expansion in Eq.(39) we obtain the renormalized potential up to two loops. We recall that

$$U(\;) = \int \mathrm{d}z S^{\bar a}(x,z) S^{\bar a}(z,y) = \frac{\partial S^{\bar a}(x,y)}{\partial m}.$$

Obviously, it is much simpler to renormalize directly from the Dyson relation. However, this does not renormalized all Feynman diagrams. Only the sum of them is finite, as was discussed in [64]. In practice, all the Feynman diagrams of a certain order cannot always be calculated, and it is necessary to renormalize each diagram separately.

9.5 Example 2: the self-energy

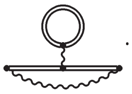
For QED in an external field, the bare and renormalized self-energies are defined by

$$S^{-1} = i\hbar c\gamma\cdot\partial - m_0 - e_0\gamma\cdot A - \Sigma,$$
$$\bar S^{-1} = i\hbar c\gamma\cdot\partial - m - e\gamma\cdot\bar A - \bar\Sigma.$$

From Dyson's relations between S and $\bar S$ we obtain the following relation between Σ and $\bar\Sigma$.

$$\bar\Sigma(e\bar a;e,m) = Z_2\Sigma(e\bar a/Z_3;e/\sqrt{Z_3},m+\delta m) + Z_2\delta m$$
$$+(Z_2-1)(-i\hbar c\gamma\cdot\partial + m + e\gamma\cdot\bar A).$$

In the self-energy, the only new diagram with respect to vacuum QED is



This diagram is not 1PI. In vacuum QED, all self-energy diagrams are 1PI, but this is not the case in the presence of an external field. This diagram is easily renormalized as a product of two 1PI diagrams:

$$R(\;) = U(\;) + eC_4(\;)U(\;)$$

$$+C_1(\;)U(\;) + eC_1(\;)C_4(\;)U(\;).$$

10 Self-consistent field renormalization

The case of self-consistent QED is simpler, since we have the same diagrams as for vacuum QED (with S^0 replaced by $S^{\bar A}$). Therefore, the renormalization rules are the same, except that the free fermion lines become the fermion lines in the SCF potential $\bar A(x)$.

Since there is no tadpole in the fermion propagator of self-consistent QED, the only point which is delicate is the renormalization of tadpoles to calculate the self-consistent potential. When all subdivergences are removed, the superficial counterterm is obtained by expanding the SCF potential in terms of strong field diagrams and by renormalizing those. The result is

$$R^\mu(x;\Gamma) = \bar R^\mu(x;\Gamma) + eC_4(\Gamma)\bar A^\mu(x).$$

Thus, the counterterm is proportional to the renormalized potential itself.

11 Renormalization conditions

The renormalization theory is a rigorous method to remove the subdivergences of a Feynman diagram. However, once all the subdivergences are removed, we must still specify the value of the superficial divergence. This is done by giving renormalization conditions. In the case of QED without external field, these conditions are well-known ([10] p. 413). However, for QED with external field, they have never been stated precisely. It is the lack of proper renormalization conditions that yields the ambiguities in the results of Dosch and Müller [52]. There are three kinds of renormalization conditions: for the current, for the photon propagator and for the electron propagator. Some of them were investigated in the early days of QED [34].

11.1 The current

The renormalization condition for the current is deduced from the neutrality of matter. The total charge of a piece of matter is the sum of the proton charges and the electron charges. Thus, the induced charge density due to the vacuum polarization integrates to zero. If this were not true, a piece of matter with an equal number of electrons and protons would have a net charge.

In the case of strong field QED, we operate the renormalization rule for tadpoles (55) with ${D^0}^{-1}_{\mu\nu}(x,y)$ and we obtain

$$\int \mathrm{d}y {D^0}^{-1}_{\mu\nu}(x,y) R(\Gamma;y,\nu) = \int \mathrm{d}y {D^0}^{-1}_{\mu\nu}(x,y)\bar{R}(\Gamma;y,\nu) + eC_4(\Gamma)\bar{\jmath}_\mu(x).$$

The left hand side represents the renormalized current induced by the external current $\bar{\jmath}_\mu(x)$. Two points must be noticed here. Firstly, the superficial divergence of the current diagrams (tadpoles) are removed by a counterterm proportional to the external current, and any induced current proportional to the external current cannot be distinguished from a renormalization of the charge (see [68] p.553). With some hindsight, this point can be recognized in Ref.[69]. Secondly, the integral of the induced current must be zero to ensure matter neutrality. Thus

$$\int \mathrm{d}x \int \mathrm{d}y {D^0}^{-1}_{\mu\nu}(x,y) R(\Gamma;y,\nu) = 0$$

This second point was clearly made by Uehling [34]. When the external current is not neutral, this is enough to determine $C_4(\Gamma)$.

In the case of self-consistent QED, the counterterm is proportional to $\bar{A}(x)$, and for a neutral atom the corresponding $\bar{J}(x)$ integrates to zero. Therefore, the criterium of matter neutrality is not enough to determine the renormalized current of self-consistent QED.

11.2 The photon Green function

To specify a renormalization condition for the photon Green function, we use the requirement that, very far from the external field, low frequency Compton scattering should be given by the Thomson formula [68] p.640 and two close charges should interact with a Coulomb potential [68] p.670 and [10] p.325.

In other words, for very large $\mathbf{x}$ and $\mathbf{y}$ and small $\mathbf{x}-\mathbf{y}$, $\bar{D}_{\mu\nu}(x,y)$ should tend to $D^0_{\mu\nu}(x,y)$.

11.3 The electron Green function

For the electron Green function, the renormalization conditions do not seem to have been studied beyond vacuum QED. One could probably assume that, for very large $\mathbf{x}$ and $\mathbf{y}$ and small $\mathbf{x}-\mathbf{y}$, $\bar{S}(x,y)$ should tend to $S^{\bar{A}}(x,y)$. The validity of these renormalization conditions should be studied in the specific case of atomic physics.

12 Conclusion

In this paper, the equations of self-consistent quantum electrodynamics were derived from the Schwinger approach. These equations are valid for any number of electrons in the system and the functional derivative of the Green function in the presence of bound electrons was used to calculate various perturbative solutions of the Schwinger equations.

An interesting application of self-consistent QED is to see if the solitonic solutions of the Maxwell-Dirac equations [70] survive when the two loop interaction is turned on. In other words, can an interacting electron be bound by its own vacuum polarization ?

The general renormalization rules for strong field QED and self-consistent QED were given. Again, these rules are valid also in the presence of bound electrons, because the bound states do not modify the short-distance or large-momentum asymptotics of the Green functions.

Much work remains to be done to transform QED into a practical tool for solid-state or molecular calculations. We must investigate more fully the conditions of renormalization. We should also be able to calculate the energy-momentum tensor from the Green functions, as the current was written in terms of the electron propagator. Results in this direction were obtained by Engel and Dreizler [1], but we have not yet reached an all-order renormalization of the QED energy. The energy is not the only physically interesting property and the investigation of the full energy-momentum tensor might be interesting and might provide some physically reasonable conditions of renormalization.

In this paper, we have only considered a coherent external source $j_\mu(x)$. It is also possible to derive Schwinger equations for a partially coherent source. The main application of this extension is the effect of temperature on an electronic system.

Finally, it would be interesting for the spectroscopic applications to generalize the Schwinger equations to a degenerate unperturbed ground state. When the unperturbed state is degenerate, we must consider not only a single matrix element $\langle\Phi|A_\mu(x)|\Phi\rangle$ but a matrix with matrix elements $\langle\Phi_i|A_\mu(x)|\Phi_j\rangle$.

13 Acknowledgements

It is a pleasure to thank Alessandra Frabetti, Eric-Olivier Le Bigot and Guy Bonneau for their thorough reading of the manuscript. I also thank Paul Indelicato for helpful discussions. This is IPGP contribution #1814.

14 Appendix

In this appendix, we give the proof of equations that are used in the text.

14.1 Derivative of $S^0_N(A)$

We saw in section 5 that $S^0_N(A) = S^0_0(A) + P_N$ with

$$P_N(x,y) = \frac{i}{\hbar c} \sum_{0<E_n\le E_F} \psi_n(x)\bar\psi_n(y).$$

For notational convenience, we suppress the argument A of the Green functions S. In this equation, the sum is over N states. The wavefunctions $\psi_n(x)$ are solutions of the Dirac equation in the presence of $A_\mu(\mathbf{x})$:

$$\big(i\hbar c\gamma\cdot\partial - mc^2 - e\gamma\cdot A(\mathbf{x})\big)\psi_n(x) = 0.$$

Therefore

$$\big(i\hbar c\gamma\cdot\partial - mc^2 - e\gamma\cdot A(\mathbf{x})\big)P_N(x,y) = 0,$$

and the adjoint equation, deduced from Eq.(19)

$$\big(-i\hbar c\partial^\mu - eA^\mu(\mathbf{x})\big)P_N(y,x)\gamma_\mu - mc^2 P_N(y,x) = 0.$$

We perturb the vector potential into $A_\mu(\mathbf{x}) + \epsilon V_\mu(x)$ so that P_N becomes $P_N + \epsilon\delta P_N$, which must be a solution of the Dirac equation for the perturbed potential. Keeping only the terms linear in ϵ we obtain

$$\big(i\hbar c\gamma\cdot\partial - mc^2 - e\gamma\cdot A(\mathbf{x})\big)\delta P_N(x,y) = e\gamma\cdot V(x)P_N(x,y).$$

The general solution of this equation is

$$\delta P_N = eS^0_0(A)\gamma\cdot V P_N + R_N,$$

where R_N is a solution of the unperturbed Dirac equation. We repeat the argument for the adjoint equation and we obtain

$$\delta P_N = eS^0_0(A)\gamma\cdot V P_N + eP_N\gamma\cdot V S^0_0(A) + R'_N,$$

where R'_N is a solution of the Dirac and adjoint Dirac equations. Finally, we calculate the functional derivative by taking $V = \delta_{\lambda,\mu}\delta(x-z)$ so that

$$\begin{aligned}\frac{\delta P_N(x,y)}{e\delta A_\lambda(z)} &= S^0_0(x,z)\gamma^\lambda P_N(z,y)\\ &\quad + P_N(x,z)\gamma^\lambda S^0_0(z,y) + R'_N.\end{aligned}$$

To determine R'_N we require that the perturbation does not change the number of bound electrons. In other words

$$\Delta N = \int \mathrm{d}\mathbf{x}\,\mathrm{tr}\Big[\gamma^0\frac{\delta P_N(x,x)}{\delta A_\lambda(y)}\Big] = 0.$$

To evaluate this, we first calculate

$$\int \mathrm{d}\mathbf{x}\,\mathrm{tr}[\gamma^0 S^0_0(x,y)\gamma^\lambda P_N(y,x)].$$

We use Eq.(24) for $E_F = 0$, because of the cyclic property of the trace we can bring the last $\bar\psi_m(x)$ in the front. then we use the orthogonality of the solutions of the Dirac equation and we find

$$\begin{aligned}\int \mathrm{d}\mathbf{x}\,\mathrm{tr}[\gamma^0 S^0_0(x,y)\gamma^\lambda P_N(y,x)] &= \theta(x^0-y^0)\\ &\times \sum_{0<E_m\le E_F} \mathrm{tr}[\bar\psi_m(y)\gamma^\lambda\psi_m(y)].\end{aligned}$$

The same equation for the second part ($P_N S^0_0$) gives the same expression, where $\theta(x^0-y^0)$ is replaced by $\theta(y^0-x^0)$. Thus

$$\begin{aligned}\int \mathrm{d}\mathbf{x}\,\mathrm{tr}[\gamma^0(S^0_0(x,y)\gamma^\lambda P_N(y,x) + P_N(x,y)\gamma^\lambda S^0_0(y,x))]\\ = \sum_{0<E_m\le E_F} \mathrm{tr}[\bar\psi_m(y)\gamma^\lambda\psi_m(y)].\end{aligned}$$

Now if we take $R'_N(x,y;z,\lambda) = P_N(x,z)\gamma^\lambda P_N(z,y)$ we obtain

$$\int \mathrm{d}\mathbf{x}\mathrm{tr}[\gamma^0 R'_N(x,x;y,\lambda)] = -\sum_{0<E_m\le E_F} \mathrm{tr}[\bar\psi_m(y)\gamma^\lambda\psi_m(y)].$$

which compensates exactly for the previous term. Finally, since we know that

$$\frac{\delta S^0_0(x,y;A)}{\delta A_\lambda(z)} = S^0_0(x,z;A)\gamma^\lambda S^0_0(z,y;A).$$

we have obtained that

$$\frac{\delta S^0_N(x,y;A)}{\delta A_\lambda(z)} = S^0_N(x,z;A)\gamma^\lambda S^0_N(z,y;A).$$

This is a very satisfactory result, which shows that the perturbative solution of the Schwinger equation has the same form for the vacuum and the charged cases.

14.2 Integral

In this section we show the following identity

$$F(\varphi) = F(0) + \int_0^1 \mathrm{d}\lambda \int \mathrm{d}x \frac{\delta F(\lambda\varphi)}{\delta\varphi(x)}\varphi(x). \tag{56}$$

This is a functional form of the classical Taylor formula

$$f(x) = f(0) + \int_0^1 \mathrm{d}\lambda f'(\lambda t)t, \tag{57}$$

where $f'(t)$ is the derivative of $f(t)$. The n-dimensional generalization of Eq.(57) is

$$\begin{aligned}f(\mathbf{x}) &= f(0) + \int_0^1 \mathrm{d}\lambda \frac{\mathrm{d}}{\mathrm{d}\lambda} f(\lambda\mathbf{x})\\ &= f(0) + \int_0^1 \mathrm{d}\lambda \sum_{i=1}^n \partial_i f(\lambda\mathbf{x})x^i. \end{aligned} \tag{58}$$

Eq.(56) can be derived as the above n-dimensional case, but we choose to prove it by an explicit calculation which shows that the purpose of the integral over λ is to change the multiplicity of some terms. We start from the Taylor expansion for functional derivatives

$$F(\varphi) = F(0) + \sum_{n=1}^{\infty} \frac{1}{n!} \int \mathrm{d}y_1 \cdots \mathrm{d}y_n \frac{\delta^n F(0)}{\delta\varphi(y_n)\cdots\delta\varphi(y_1)} \times \varphi(y_1)\cdots\varphi(y_n).$$

The notation used in this equation means that the functional derivatives of F are taken at $\phi = 0$. Therefore, using the symmetry of functional derivatives with respect to their arguments

$$\frac{\delta F(\varphi)}{\delta\varphi(x)} = \sum_{n=1}^{\infty} \frac{1}{(n-1)!} \int \mathrm{d}y_1 \cdots \mathrm{d}y_{n-1} \times \frac{\delta^n F(0)}{\delta\varphi(x)\delta\varphi(y_{n-1})\cdots\delta\varphi(y_1)} \varphi(y_1)\cdots\varphi(y_{n-1}).$$

In Eq.(56) the purpose of the integral over λ is to replace the factor $1/(n-1)!$ by the correct factor $1/n!$.

$$\frac{\delta F(\lambda\varphi)}{\delta\varphi(x)} = \sum_{n=1}^{\infty} \frac{\lambda^{n-1}}{(n-1)!} \int \mathrm{d}y_1 \cdots \mathrm{d}y_{n-1} \times \frac{\delta^n F(0)}{\delta\varphi(x)\delta\varphi(y_{n-1})\cdots\delta\varphi(y_1)} \varphi(y_1)\cdots\varphi(y_{n-1}).$$

The notation $\delta F(\lambda\varphi)/\delta\varphi(x)$ has the following meaning. The function $G(x,\varphi) = \delta F(\varphi)/\delta\varphi(x)$ is a function of two independent variables (x and φ). Then $\delta F(\lambda\varphi)/\delta\varphi(x) = G(x, \lambda\varphi)$.

Therefore,

$$\begin{aligned}\int_0^1 \mathrm{d}\lambda \int \mathrm{d}x \frac{\delta F(\lambda\varphi)}{\delta\varphi(x)} \varphi(x) &= \sum_{n=1}^{\infty} \frac{1}{n!} \int \mathrm{d}y_1 \cdots \mathrm{d}y_{n-1}\mathrm{d}x \\ &\times \frac{\delta^n F(0)}{\delta\varphi(x)\delta\varphi(y_{n-1})\cdots\delta\varphi(y_1)} \\ &\times \varphi(y_1)\cdots\varphi(y_{n-1})\varphi(x) \\ &= F(\varphi) - F(0).\end{aligned}$$

Note that an obvious consequence of Eq.(56) is that

$$\frac{\delta}{\delta\varphi(x)} \int_0^1 \mathrm{d}\lambda \int \mathrm{d}x \frac{\delta F(\lambda\varphi)}{\delta\varphi(x)} \varphi(x) = \frac{\delta F(\varphi)}{\delta\varphi(x)}. \quad (59)$$

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