Quantum field theory of degenerate systems

Christian Brouder
Laboratoire de Minéralogie Cristallographie, CNRS UMR 7590, Universités Paris 6 et 7, IPGP, case 115, 4 place Jussieu, 75252 Paris cedex 05, France.
(Dated: July 22, 2003)

To set up a self-consistent quantum field theory of degenerate systems, the unperturbed state should be described by a density matrix instead of a pure state. This increases the combinatorial complexity of the many-body equations. Hopf algebraic techniques are used to deal with this complexity and show that the Schwinger-Dyson equations are modified in a non-trivial way. The hierarchy of Green functions is derived for degenerate systems, and the case of a single electron in a two-fold degenerate orbital is calculated in detail.

The degeneracy of quantum systems is the basis of important physical phenomena such as superconductivity, magnetism, Bose-Einstein condensation, Jahn-Teller distortion, etc. Degenerate systems are also numerous. For instance, all systems containing an odd number of electrons are degenerate by Kramers' theorem [1]. Moreover, degenerate systems are a subject of active research within the density functional approach [2–5].

The quantum field theory of degenerate (or quasidegenerate) systems has a long history [6–9] and developed recently into the two-time Green function method [10, 11] and the covariant-evolution-operator method [12, 13]. Still, these theories have a severe drawback: either they are not self-consistent or they do not preserve the symmetry of the system. The antagonism between self-consistency and symmetry can be seen on a simple example. If the Hamiltonian of a system is spherically symmetric, the energy eigenstates can be classified with angular momentum quantum numbers $|LM\rangle$. However, if $L \geq 1$, the system is degenerate and no state $|LM\rangle$ gives a spherically symmetric charge density [14]. So, if that charge density is used to construct a self-consistent potential, spherical symmetry is broken. The only way to recover the symmetry is to describe the system with the density matrix $(2L+1)^{-1}\sum_M |LM\rangle\langle LM|$. In other words, a self-consistent theory of degenerate systems must use a density matrix instead of a pure state.

In this paper, we set up a quantum field theory of degenerate systems based on density matrices. This is a non trivial extension of the nondegenerate theory because the usual sum over occupied states must be modified, and the hierarchy of Green functions acquires additional terms that increase considerably the combinatorial complexity of the calculations. The problem is solved with two technical tools: the nonequilibrium quantum field theory and the Hopf algebra of derivations.

The generating function – In nonequilibrium quantum field theory [15, 16], the Green functions can be obtained by the functional derivative of a generating function Z_ρ with respect to double external sources (this is equivalent to Keldysh' closed-time-path method [17]). In this paper we consider nonrelativistic many-body theory and the external sources $\eta_+, \eta_-, \bar\eta_+, \bar\eta_-$ are fermionic. The density matrix is $\hat\rho = \sum_{KL} \rho_{LK} |L\rangle\langle K|$ and the generating function is

$$Z_\rho \;=\; \mathrm{tr}\big(\hat\rho S(\bar\eta_-,\eta_-)^{-1} S(\bar\eta_+,\eta_+)\big), \qquad (1)$$

where $S(\bar\eta_\pm,\eta_\pm) = T\exp\big(-i\int_{-\infty}^{\infty} H^{\mathrm{int}}(t)\mathrm{d}t + i\int \bar\eta_\pm(x)\psi(x) + \psi^\dagger(x)\eta_\pm(x)\mathrm{d}x\big)$, is the S-matrix, $\psi(x)$ is the electron field operator and $H^{\mathrm{int}}(t)$ is the interacting Hamiltonian. According to the functional formulation of nonequilibrium QFT [16], the generating function can be written $Z_\rho = \mathrm{e}^{-iD} Z^0_\rho$, where $D = D_+ - D_-$ and $D_\pm$ is the interaction term $\int_{-\infty}^{\infty} H^{\mathrm{int}}(t)\mathrm{d}t$ where the fields are replaced by functional derivatives with respect to the external sources $\bar\eta_\pm, \eta_\pm$. Finally, Z^0_ρ is the generating function for the noninteracting system: $Z^0_\rho = \exp[-i\int \bar{\boldsymbol\eta}(x) G^0_0(x,y)\boldsymbol\eta(y)\mathrm{d}x\mathrm{d}y]\sum_{KL}\rho_{LK}N^0_{KL}$. In this equation, $\boldsymbol\eta$ is a two-dimensional vector with components η_+ and η_-, $G^0_0(x,y)$ is the 2x2 matrix

$$\begin{pmatrix} -i\langle 0|T\big(\psi(x)\psi^\dagger(y)\big)|0\rangle & -i\langle 0|\psi^\dagger(y)\psi(x)|0\rangle \\ i\langle 0|\psi(x)\psi^\dagger(y)|0\rangle & -i\langle 0|T^*\big(\psi(x)\psi^\dagger(y)\big)|0\rangle \end{pmatrix},$$

T^* is the anti-time-ordering operator and $N^0_{KL} = \langle K|\!:\exp\big[i\int \bar\eta_d(x)\psi(x) + \psi^\dagger(x)\eta_d(x)\mathrm{d}x\big]\!:\!|L\rangle$, with $\eta_d = \eta_+ - \eta_-$. Notice that N^0_{KL} is the generating function for the matrix elements of normal products of field operators. The proof of these formulas can be found in [16]. The two technical problems that will be solved in this paper are the explicit calculation of N^0_{KL} and the manipulation of e^{-iD} to recover a hierarchy of Green functions for degenerate systems.

To calculate N^0_{KL}, we must define precisely the states $|K\rangle$ and $|L\rangle$. The solution of the noninteracting Schrödinger equation provides one-electron orbitals $u_n(\mathbf{r})$ with energy ϵ_n. An orbital is called a core orbital if it is filled in all states $|K\rangle$ and $|L\rangle$, otherwise, it is called a valence orbital. The core orbitals are numbered from 1 to C, the valence orbitals from 1 to M. There are C electrons in the core orbitals and $N < M$ electrons in the valence orbitals. For example, in the ion Cr^{3+}, there are $C = 18$ core orbitals and $N = 3$ electrons in the $M = 10$ orbitals of the degenerate 3d shell. The states are generated from the vacuum $|0\rangle$ by the action of creation operators $c^\dagger_n$ for core electrons and $v^\dagger_n$ for valence electrons as

$$|K\rangle = v^\dagger_{i_N}\dots v^\dagger_{i_1} c^\dagger_C \dots c^\dagger_1|0\rangle,$$
$$|L\rangle = v^\dagger_{j_N}\dots v^\dagger_{j_1} c^\dagger_C \dots c^\dagger_1|0\rangle,$$

where i_k, j_k are valence orbitals (i.e. integers taken in the set $\{1,\dots,M\}$) ordered so that $i_1 < \dots < i_N$ and $j_1 < \dots < j_N$. A lengthy calculation yields

$$N^0_{KL} = \prod_{k=1}^{C}(1+\bar\alpha_k\alpha_k) \times \exp\Big(\sum_{n=1}^{M}\frac{\partial^2}{\partial\alpha_n\partial\bar\alpha_n}\Big)\bar\alpha_{j_1}\alpha_{i_1}\dots\bar\alpha_{j_N}\alpha_{i_N},$$

with $\bar\alpha_n = \int\bar\eta_d(x)u_n(x)\mathrm{d}x$ and $\alpha_n = \int u^\dagger_n(x)\eta_d(x)\mathrm{d}x$, where $u_n(x) = \mathrm{e}^{-i\epsilon_n t}u_n(\mathbf{r})$, $u^\dagger_n(x) = \mathrm{e}^{i\epsilon_n t}u^\dagger_n(\mathbf{r})$ and $x=(t,\mathbf{r})$. Notice that α_n and $\bar\alpha_n$ are anticommuting (i.e. Grassmann) variables because the sources are anticommuting variables.

For the following, we shall need the quantity

$$W^0_\rho = \log Z^0_\rho = -i\int\bar{\boldsymbol\eta}(x)G^0_0(x,y)\boldsymbol\eta(y)\mathrm{d}x\mathrm{d}y + \sum_{k=1}^{C}\bar\alpha_k\alpha_k + \log(\rho(\bar\alpha,\alpha)), \quad (2)$$

where

$$\rho(\bar\alpha,\alpha) = \exp(\sum_{n=1}^{M}\frac{\partial^2}{\partial\alpha_n\partial\bar\alpha_n})\sum_{KL}\rho_{LK}\bar\alpha_{j_1}\alpha_{i_1}\dots\bar\alpha_{j_N}\alpha_{i_N}.$$

To derive (2), we used the fact that $\log(1+\bar\alpha_k\alpha_k) = \bar\alpha_k\alpha_k$, because $\bar\alpha_k$ and α_k are anticommuting variables. For the same reason, $\log(\rho(\bar\alpha,\alpha))$ is a finite polynomial in $\bar\alpha$ and α. The term $-i\int\bar{\boldsymbol\eta}(x)G^0_0(x,y)\boldsymbol\eta(y)\mathrm{d}x\mathrm{d}y$ is linear in $\eta_\pm$ and $\bar\eta_\pm$. The sum over core orbitals is also linear in $\eta_\pm$ and $\bar\eta_\pm$. So we write $\log(\rho(\bar\alpha,\alpha)) = \rho_l(\bar\alpha,\alpha) + \rho_c(\bar\alpha,\alpha)$, where $\rho_l(\bar\alpha,\alpha)$ gathers the terms of $\log(\rho(\bar\alpha,\alpha))$ which are linear in $\eta_\pm$ and $\bar\eta_\pm$, and we obtain $W^0_\rho = -i\int\bar{\boldsymbol\eta}(x)G^0_\rho(x,y)\boldsymbol\eta(y)\mathrm{d}x\mathrm{d}y + \rho_c(\bar\alpha,\alpha)$, where $G^0_\rho(x,y)$ is the sum of all the terms that are linear in $\eta_\pm$ and $\bar\eta_\pm$.

In the standard case of a nondegenerate system [18], the electron-hole transformation shows that the free Green function $G^0(x,x')$ is a sum over occupied states for $t'>t$ and over empty states for $t<t'$. However, this transformation is usually justified at the first order only, by showing that it modifies the free Hamiltonian by a constant [18]. Equation (2) shows that the electron-hole transformation is valid at all orders. In the degenerate case, additional terms come from $\rho_l(\bar\alpha,\alpha)$.

The Hopf algebra of derivations – The hierarchy of Green functions will be obtained through a manipulation of the exponential of functional derivatives e^{-iD}. This manipulation will be made with the help of the Hopf superalgebra [19] of derivations. We call derivation a functional derivative with respect to $\eta_\pm(x)$ or $\bar\eta_\pm(x)$. We start from the vector space V generated by the basis elements $\delta/\delta\eta_\pm(x)$ and $\delta/\delta\bar\eta_\pm(x)$. The exterior Hopf algebra $\Lambda(V)$ is then a standard mathematical object [20] and we recall here its main properties. The product is the composition of functional derivatives (e.g. the product of $\delta/\delta\eta_+(x)$ and $\delta/\delta\bar\eta_-(y)$ is $\delta^2/\delta\eta_+(x)\delta\bar\eta_-(y)$). This product is anticommutative. If an element D of $\Lambda(V)$ is a product of n derivations, we say that its degree is $\deg(D)=n$ and its parity is $|D|=0$ if n is even and $|D|=1$ if n is odd. An element D is called even (resp. odd) if $|D|=0$ (resp. $|D|=1$). The product satisfies the identity $DD' = (-1)^{|D||D'|}D'D$. Thus, an even element D commutes with all elements of $\Lambda(V)$. We also introduce a unit $\mathbf{1}$ such that $\mathbf{1}D = D\mathbf{1} = D$ for all $D\in\Lambda(V)$.

The main operation that we shall use is the coproduct. It is a linear map Δ from $\Lambda(V)$ to $\Lambda(V)\otimes\Lambda(V)$ such that $\Delta\mathbf{1} = \mathbf{1}\otimes\mathbf{1}$ and $\Delta\partial = \partial\otimes\mathbf{1} + \mathbf{1}\otimes\partial$ if ∂ is a derivation. To define the coproduct of a term D of degree greater than 1, we use Sweedler's notation for the coproduct: $\Delta D = \sum D_{(1)}\otimes D_{(2)}$ and the recursive definition

$$\Delta(DD') = \sum(-1)^{|D_{(2)}||D'_{(1)}|}(D_{(1)}D'_{(1)})\otimes(D_{(2)}D'_{(2)}).$$

For example, if ∂ and ∂' are derivations,

$$\Delta(\partial\partial') = \partial\partial'\otimes 1 + 1\otimes\partial\partial' + \partial\otimes\partial' - \partial'\otimes\partial.$$

The convenience of the coproduct stems from the basic identity

$$D(fg) = \sum(-1)^{|D_{(2)}||f|}(D_{(1)}f)(D_{(2)}g), \quad (3)$$

where f and g are functions of the external sources and the fermion fields, $\deg(f)=n$ if f is a product of n sources or fields and $|f| = \deg(f)$ modulo 2.

The hierarchy of Green functions will be derived from an identity involving the reduced coproduct with respect to D, denoted by Δ'. It is defined by $\Delta'\mathbf{1} = \mathbf{1}\otimes\mathbf{1}$, $\Delta' D = \Delta D - \mathbf{1}\otimes D - D\otimes\mathbf{1}$ and then recursively by

$$\Delta'(D^{n+1}) = \sum(-1)^{|D_{(1')}||D^n_{(2')}|}D^n_{(1')}D_{(1')}\otimes D^n_{(2')}D_{(2')}.$$

The main property of this reduced coproduct is the following identity, valid for any even element D of $\Lambda(V)$.

$$\Delta\mathrm{e}^{-iD} = (\Delta'\mathrm{e}^{-iD})(\mathrm{e}^{-iD}\otimes\mathrm{e}^{-iD}). \quad (4)$$

The hierarchy of Green functions – The one-body Green function is $G(x,y) = (1/Z_\rho)\partial'\partial Z_\rho|_{\eta=\bar\eta=0}$, where $\partial = i\delta/\delta\bar\eta_\pm(x)$ and $\partial' = \delta/\delta\eta_\pm(y)$. The definition (1) of Z_ρ, and the fact that $\mathrm{tr}(\hat\rho)=1$ yield $Z_\rho|_{\eta=\bar\eta=0}=1$. From $Z_\rho = \mathrm{e}^{-iD}Z^0_\rho$ and $Z_\rho = \mathrm{e}^{W^0_\rho}$ we obtain $\partial Z_\rho = \mathrm{e}^{-iD}(\partial Z^0_\rho) = \mathrm{e}^{-iD}(\partial W^0_\rho)Z^0_\rho$. Noticing that W^0_ρ is even, we operate now with ∂' and obtain

$$G(x,y) = \mathrm{e}^{-iD}\big((\partial'\partial W^0_\rho)Z^0_\rho\big) - \mathrm{e}^{-iD}\big((\partial W^0_\rho)(\partial' Z^0_\rho)\big),$$

all quantities being evaluated at $\boldsymbol{\eta} = \bar{\boldsymbol{\eta}} = 0$. Now we apply equations (3) and (4) and we obtain

$$\begin{aligned} G(x,y) &= \sum (\mathrm{e}^{-iD})_{(1')} (\partial' \partial W^1_\rho)(\mathrm{e}^{-iD})_{(2')} Z_\rho \\ &\quad - \sum (\mathrm{e}^{-iD})_{(1')} (\partial W^1_\rho)(\mathrm{e}^{-iD})_{(2')} (\partial' Z_\rho), \end{aligned}$$

where $W^1_\rho = \mathrm{e}^{-iD} W^0_\rho$. The exponential can be expanded

$$\begin{aligned} G(x,y) &= \sum_{n=0}^{2M-2} \frac{(-i)^n}{n!} \sum \big(D^n_{(1')} \partial' \partial W^1_\rho\big)\big(D^n_{(2')} Z_\rho\big) \\ &- \sum_{n=0}^{2M-1} \frac{(-i)^n}{n!} \sum (-1)^{|D^n_{(2')}|} \big(D^n_{(1')} \partial W^1_\rho\big)\big(D^n_{(2')} \partial' Z_\rho\big). \quad (5) \end{aligned}$$

Equation (5) is the Schwinger-Dyson equation for degenerate systems. It is the main result of the paper. By further differentiating with respect to external sources and using (3), we obtain the hierarchy of the n-body Green functions. The number of terms of equation (5) is finite, but it can be large when M is large. However, the recursive nature of the coproduct makes it well suited for symbolic-algebra packages. Notice that equation (5) has to be used not only for degenerate systems, but more generally when the unperturbed state cannot be represented by a single Slater determinant.

The simplest example– As an illustration, we treat in detail the simplest case of a single electron in a two-fold degenerate level (i.e. $N = 1$ and $M = 2$). This example is important because it corresponds to Kramers' degeneracy. The Hamiltonian of a nonrelativistic quantum system is

$$\begin{aligned} H &= \int \psi^\dagger(\mathbf{r})\Big(-\frac{\Delta}{2m} + U_n(\mathbf{r})\Big)\psi(\mathbf{r})\mathrm{d}\mathbf{r} \\ &\quad + \frac{e^2}{2} \int \psi^\dagger(\mathbf{r})\psi^\dagger(\mathbf{r}') \frac{1}{|\mathbf{r}-\mathbf{r}'|} \psi(\mathbf{r}')\psi(\mathbf{r})\mathrm{d}\mathbf{r}\mathrm{d}\mathbf{r}', \quad (6) \end{aligned}$$

where $U_n(\mathbf{r})$ is the nuclear potential. This Hamiltonian will be split into two parts as $H = H_0 + H_1$, where H_0 is the first term of H in (6) and H_1 its second term. Thus the interaction is described by the differential operator $D = D_+ - D_-$ with

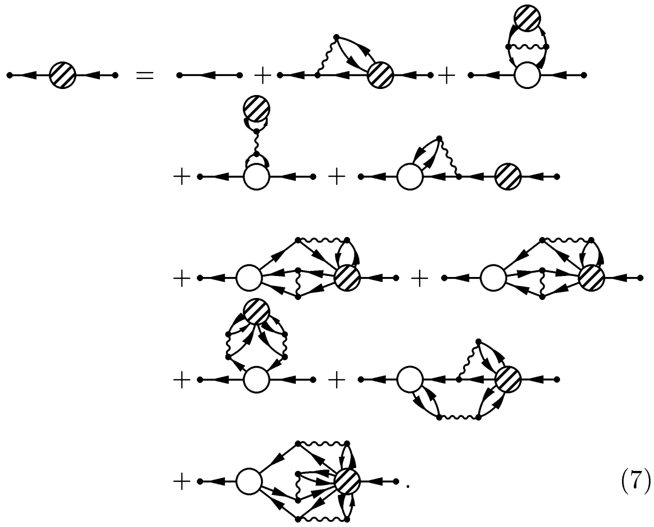
$$\begin{aligned} D_\pm &= \frac{e^2}{2} \int \mathrm{d}t \mathrm{d}\mathbf{r} \mathrm{d}\mathbf{r}' \frac{1}{|\mathbf{r}-\mathbf{r}'|} \\ &\qquad \frac{\delta^4}{\delta\eta_\pm(t,\mathbf{r})\delta\eta_\pm(t,\mathbf{r}')\delta\bar{\eta}_\pm(t,\mathbf{r}')\delta\bar{\eta}_\pm(t,\mathbf{r})}. \end{aligned}$$

From this definition we obtain $W^1_\rho = \mathrm{e}^{-iD} W^0_\rho = W^0_\rho$.

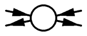
The degenerate levels will be called $n = 1$ and $n = 2$. The function $\rho(\bar{\alpha},\alpha)$ is $\rho(\bar{\alpha},\alpha) = 1 + (1/2)(\bar{\alpha}_1\alpha_1 + \bar{\alpha}_2\alpha_2)$. Thus $\log(\rho(\bar{\alpha},\alpha)) = (1/2)(\bar{\alpha}_1\alpha_1 + \bar{\alpha}_2\alpha_2) - (1/4)\bar{\alpha}_1\alpha_1\bar{\alpha}_2\alpha_2$. The first term is added to the free Green function by defining

$$G^0_\rho(x,y) = G^0_0(x,y) + i\rho_l(x,y) \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix},$$

with $\rho_l(x,y) = (1/2)(u_1(x)u_1^\dagger(y) + u_2(x)u_2^\dagger(y))$, the second term is $\rho_c(\bar{\alpha},\alpha) = -(1/4)\bar{\alpha}_1\alpha_1\bar{\alpha}_2\alpha_2$. Equation (5) is now rewritten in terms of Feynman diagrams:



(7)

In this equation, the interacting 1-body Green function is $G(x,y) =$  and the same dot with $2n$ fermion lines is the interacting n-body Green function, the free Green function is $G^0_\rho(x,y) =$, the Coulomb interaction term is , and $r(x_1,x_2,y_1,y_2) =$  describes the correlation induced by the degeneracy of the system:

$$\begin{aligned} r(x_1,x_2,y_1,y_2) &= -\frac{\delta^4 W^1_\rho}{\delta\bar{\eta}_\pm(x_1)\delta\bar{\eta}_\pm(x_2)\delta\eta_\pm(y_1)\delta\eta_\pm(y_2)} \\ &= (-1/4)(u_1(x_1)u_2(x_2) - u_2(x_1)u_1(x_2)) \\ &\quad (u_1^\dagger(y_1)u_2^\dagger(y_2) - u_2^\dagger(y_1)u_1^\dagger(y_2)). \end{aligned}$$

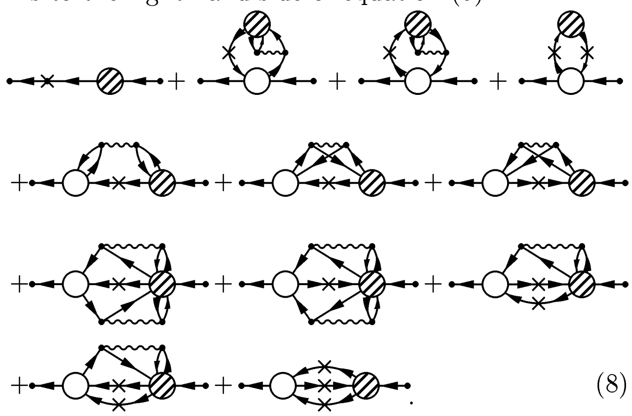
The first term of (7) corresponds to $n = 0$ in equation (5), the next four terms are for $n = 1$, the next four terms for $n = 2$ and the last term for $n = 3$. The first term and the terms where the initial and final points are connected to the white dot (i.e. 3, 4 and 8) come from the first line of equation (5), the terms 2, 5, 6, 7, 9 and 10 come from the second line. For a nondegenerate system, only the first two terms of the right hand side survive. So we see that degeneracy adds a large number of terms to the hierarchy. It should be noticed that if equation (7) is operated by $i\partial/\partial t - h_0$ (where $h_0 = -\Delta/2m + U_n$), the first term of the right hand side gives $\delta(x - x')$, the second term remains and all the other terms vanish because $(i\partial/\partial t - h_0)u_i(x) = 0$. In other words, the differential form of the hierarchy is the same for degenerate and nondegenerate systems, but the integral form (7) is different. So the additional terms can be seen as boundary conditions due to the correlation of the system induced by its degeneracy. Finally, it should be stressed that this hierarchy is non perturbative. All Green functions involved in equation (7) are interacting. However, this hierarchy can be used to derive a perturbative expansion of the Green function (see, e.g. [21]).

To solve equation (7), the hierarchy must be closed by using one of the standard approximations such as the random-phase approximation, the GW-approximation or the Bethe-Salpeter equation.

Self-consistency – The previous equations were not self-consistent. To obtain self-consistent equations the Hamiltonian H will be split into two parts as $H = H_0 + H_1$, with $H_0 = \int \psi^\dagger(\mathbf{r})(-\frac{\Delta}{2m} + U_n(\mathbf{r}) - V(\mathbf{r}))\psi(\mathbf{r})\mathrm{d}\mathbf{r}$ and $H_1 = H - H_0$. The potential $V(\mathbf{r})$ will be determined self-consistently. The presence of this external field adds the term $D^v_\pm$ to $D_\pm$, where

$$D^v_\pm = \int \mathrm{d}t\mathrm{d}\mathbf{r} V(\mathbf{r}) \frac{\delta^2}{\delta\eta_\pm(t,\mathbf{r})\delta\bar{\eta}_\pm(t,\mathbf{r})},$$

and the differential operator describing the interaction is now $D^{\mathrm{SCF}} = D_+ + D^v_+ - D_- - D^v_-$. This new interaction does not modify $W^1_\rho = W^0_\rho$, but it adds the following terms to the right-hand side of equation (5):



(8)

The condition of self-consistency is now that the potential $V(\mathbf{r})$ is generated by the electron charge density of the system or $\times = -$  in terms of Feynman diagrams.

In this paper we showed how to calculate the Green function of a degenerate system, and we derived the corresponding hierarchy of Green functions. An advantage of our approach is that it preserves the symmetry of the Hamiltonian if the closure of the hierarchy is done properly. Recall that this is not possible if we use states instead of density matrices.

Equation (5) is written in terms of unconnected Green functions. By defining a second type of reduced coproduct, it is possible to write the hierarchy of connected Green functions. In the near future, several developments of the present formalism will be presented. Firstly, if we add photon fields, the same formalism can be used to derive the Schwinger-Dyson equations for the Green functions of a degenerate systems in quantum electrodynamics; this will be applied to the nonperturbative calculation of a degenerate atomic system. Secondly, in the derivation of equation (5), no special form was assumed for the density matrix $\hat{\rho}$. So the present formalism can be used also for out-of-equilibrium quantum field theory. In particular, we can calculate the energy of the interacting system corresponding to a general density matrix $\hat{\rho}$. If we minimize the total energy with respect to $\hat{\rho}$, we obtain a set of equations that unify the Green function and diagonalization methods of many-body theory.

I gratefully acknowledge the fruitful comments by Matteo Calandra, Philippe Sainctavit and Francesco Mauri.

[1] H. Kramers, Proc. Acad. Amsterdam **33**, 959 (1930).
[2] M. Levy and A. Nagy, Phys. Rev. Lett. **83**, 4361 (1999).
[3] W. Yang, Y. Zhang, and P. Ayers, Phys. Rev. Lett. **84**, 5172 (2000).
[4] C. Ullrich and W. Kohn, Phys. Rev. Lett. **87**, 093001(1 (2001).
[5] V. Sahni and X.-Y. Pan, Phys. Rev. Lett. **90**, 123001(1 (2003).
[6] C. Bloch and J. Horowitz, Nucl. Phys. **8**, 91 (1958).
[7] B. Brandow, Rev. Mod. Phys. **39**, 771 (1967).
[8] T. Kuo, S. Lee, and K. Ratcliff, Nucl. Phys. A **176**, 65 (1971).
[9] I. Lindgren and J. Morrison, *Atomic Many-Body Theory* (Springer, Berlin, 1986).
[10] E.-O. L. Bigot, P. Indelicato, and V. Shabaev, Phys. Rev. A **63**, 040501 (2001).
[11] V. Shabaev, Phys. Repts. **356**, 119 (2002).
[12] I. Lindgren, Mol. Phys. **98**, 1159 (2000).
[13] I. Lindgren, B. Åsén, S. Salomonson, and A.-M. Mårtensson-Pendrill, Phys. Rev. A **64**, 062505 (2001).
[14] E. Lieb, Int. J. Quant. Chem. **24**, 243 (1983).
[15] J. Schwinger, J. Math. Phys. **2**, 407 (1961).
[16] K.-C. Chou, Z.-B. Su, B.-L. Hao, and L. Yu, Phys. Repts. **118**, 1 (1985).
[17] L. Keldysh, Soviet Phys. JETP **20**, 1018 (1965).
[18] A. Fetter and J. Walecka, *Quantum Theory of Many-Particle Systems* (McGraw-Hill, Boston, 1971).
[19] S. Majid, *Foundations of Quantum Group Theory* (Cambridge University Press, Cambridge, 1995).
[20] D. Eisenbud, *Commutative Algebra with a view toward Algebraic Geometry* (Springer, Berlin, 1995).
[21] A. Hall, J. Phys. A: Math. Gen. **8**, 214 (1975).